16.5 (a) ~~那么~~ $J\psi|_{t=0} = (N^S(\alpha(0)), \dot\alpha(0) + s(\dot N\circ\alpha)(0))$

so $X(s) = J\psi|_{t=0}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \dot\alpha(0) + s(\dot N\circ\alpha)(0)$

$X(0) = \dot\alpha(0) = v$. $\dot X(s) = (\dot N\circ\alpha)(0)$ So $\dot X(0) = (\dot N\circ\alpha)(0) = -L_p(v)$

(b) $\ddot X(s) = 0$. So $X(s) = (\psi(s,0), \dot\alpha(0) + s(\dot N\circ\alpha)(0)) = (\beta(s), v + sw)$

(c) $X(s) = 0 \Leftrightarrow$ $v = -s\cdot(\dot N\circ\alpha)(0) = sL_p(v)$

So $\frac{1}{s}$ is a principal curvature and v is a principal curvature direction

By Thm 1, $\Leftrightarrow \alpha(0) + \frac{1}{\frac{1}{s}}\cdot N^S(\alpha(t)) = \alpha(0) + sN^S(\alpha(t)) = \beta(s)$ is focal point of S along β.

17.1 $V(\varphi) = \int_0^{2\pi}\int_0^h \left|\det\begin{vmatrix} -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & \sin\theta & 0\end{vmatrix}\right| dt\,d\theta = \int_0^{2\pi}\int_0^h r\,dt\,d\theta = 2\pi h$

17.2 $E_1 = \frac{\partial\varphi}{\partial\theta} = (-tr\sin\theta, tr\cos\theta, 0)$, $E_2 = \frac{\partial\varphi}{\partial t} = (r\cos\theta, r\sin\theta, -h)$

$E_1\cdot E_1 = t^2r^2$. $E_2\cdot E_2 = r^2 + h^2$ $E_1\cdot E_2 = 0$ $V(\varphi) = \int_0^{2\pi}\int_0^1 \sqrt{t^2r^2(r^2+h^2)}\,dt\,d\theta = \pi r\sqrt{r^2+h^2}$

17.3 $E_1 = \frac{\partial\varphi}{\partial\theta} = (-(a+b\cos\phi)\sin\theta, (a+b\cos\phi)\cos\theta, 0)$, $E_2 = \frac{\partial\varphi}{\partial\phi} = (-b\sin\phi\cos\theta, -b\sin\phi\sin\theta, +b\cos\phi)$

$E_1\cdot E_1 = (a+b\cos\phi)^2$, $E_2\cdot E_2 = b^2$, $E_1\cdot E_2 = 0$ $V(\varphi) = \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos\phi)\,d\phi\,d\theta = 4\pi^2ab$

17.4 $E_1 = \frac{\partial\varphi}{\partial\theta} = (-a\sin\theta, a\cos\theta, 0, 0)$, $E_2 = \frac{\partial\varphi}{\partial\phi} = (0, 0, -b\sin\phi, b\cos\phi)$, $E_1\cdot E_1 = a^2$, $E_2\cdot E_2 = b^2$, $E_1\cdot E_2 = 0$

$V(\varphi) = \int_0^{2\pi}\int_0^{2\pi} ab\,d\theta\,d\phi = 4\pi^2ab$

17.5 $E_1 = \frac{\partial\varphi}{\partial\phi} = (\cos\phi\sin\theta\sin\psi, -\sin\phi\sin\theta\sin\psi, 0, 0)$ $E_2 = \frac{\partial\varphi}{\partial\theta} = (\sin\phi\cos\theta\sin\psi, \cos\phi\cos\theta\sin\psi, -\sin\theta\sin\psi, 0)$

$E_3 = \frac{\partial\varphi}{\partial\psi} = (\sin\phi\sin\theta\cos\psi, \cos\phi\sin\theta\cos\psi, \cos\theta\cos\psi, -\sin\psi)$.

$E_1\cdot E_1 = (\sin\theta\sin\psi)^2$, $E_2\cdot E_2 = (\sin\psi)^2$, $E_3\cdot E_3 = 1$. $E_i\cdot E_j = 0$ $(i\neq j)$.

So $V(\varphi) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sin\theta(\sin\psi)^2\,d\phi\,d\theta\,d\psi = 2\pi^2$

17.6 $E_1 = (x'(t), y'(t)\cos\theta, y'(t)\sin\theta)$, $E_2 = (0, -y(t)\sin\theta, y(t)\cos\theta)$

$E_1\cdot E_1 = x'(t)^2 + y'(t)^2$, $E_2\cdot E_2 = y(t)^2$, $E_1\cdot E_2 = 0$.

$V(\varphi) = \int_a^b\int_0^{2\pi}\sqrt{x'(t)^2+y'(t)^2}\,y(t)\,d\theta\,dt = 2\pi\int_a^b y(t)(x'(t)^2 + y'(t)^2)^{1/2}\,dt$

17.7 Let $a_i = \frac{\partial g}{\partial u_i}$ then $E_i = \frac{\partial\varphi}{\partial u_i} = (0,\cdots 0, 1, 0, \cdots, 0, a_i)$ $N = \pm(a_1, \cdots, a_n, -1)/(1+\sum_{i=1}^n a_i^2)^{\frac12}$

As $\begin{vmatrix} 1 & & & a_1 \\ & \ddots & & \vdots \\ & & 1 & a_n \\ a_1 & \cdots & a_n & -1\end{vmatrix} = \cancel{\begin{vmatrix} 1+a_1^2 & & 0 \\ & 1+a_n^2 & 0 \\ a_1 & \cdots a_n & 1\end{vmatrix} = \prod_{i=1}^n(1+a_i^2)}$, So $= \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ & 1 & a_n \\ a_1 & a_n & -1-a_1^2\end{vmatrix} = \begin{vmatrix} 1 & & a_2 \\ & \ddots & \vdots \\ & 1 & a_n \\ a_2 & \cdots a_n & -1-a_1^2\end{vmatrix} = \cdots = -(1+\sum_{i=1}^n a_i^2)$

So $V(\varphi) = \int_U \left(1+\sum_{i=1}^{n} a_i^2\right) / \left(1+\sum_{i=1}^{n} a_i^2\right)^{1/2} = \int_U \left(1+\sum_{i=1}^{n} (\partial g/\partial u_i)^2\right)^{1/2}$.

17.8 (a) Prove $J\varphi_n$ is not singular. We prove by induction. When $n=2$. $J_2 = \begin{pmatrix} \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 & 0 \\ \sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_2 \end{pmatrix}$

rank J_2 = rank $J_2 \cdot J_2^T$ = rank $\begin{pmatrix} \sin^2\theta_2 & 0 \\ 0 & 1 \end{pmatrix} = 2$, So J_2 is fully ranked. Suppose rank $J_{n-1} = n-1$.

~~Denoted~~ Denote the $(i,j)^{th}$ component of J_n as $J_n^{i,j}$, $i=1\cdots n+1$, $j=1\cdots n$. then

$$J_n = \begin{pmatrix} J_{n-1}^{1,1}\sin\theta_n & \cdots & J_{n-1}^{1,n-1}\sin\theta_n & \varphi_{n-1}^1\cos\theta_n \\ & \ddots & J_{n-1}^{n,n-1}\sin\theta_n & \varphi_{n-1}^n\cos\theta_n \\ 0 & \cdots & 0 & -\sin\theta_n \end{pmatrix} = \begin{pmatrix} \sin\theta_n \cdot J_{n-1} & \varphi_{n-1}\cos\theta_n \\ 0 & -\sin\theta_n \end{pmatrix}$$, so rank $J_n = n-1+1 = n$. i.e., φ_n is parametrized n-surface.

(b) ~~φ_n maps U one to one onto a subset of unit n-sphere S^n: $\{(a_1\cdots a_{n+1}) \mid \sum_{i=1}^{n+1} a_i = 1\}$,~~

Let φ_n^i be the i^{th} component of φ_n. then $\sum_{i=1}^{n+1} {\varphi_n^i}^2 = 1$ So φ_n maps U to a subset of S^n.

We only need to prove one to one. If $\varphi_n(\theta_1\cdots\theta_n) = \varphi_n(\hat\theta_1,\cdots,\hat\theta_n)$, then $\cos\theta_n = \cos\hat\theta_n$, As $\theta_n, \hat\theta_n \in (0,\pi)$ so $\theta_n = \hat\theta_n$. as $\sin\hat\theta_n \neq 0$ So $\varphi_{n-1}(\theta_1,\cdots,\theta_{n-1}) = \varphi_{n-1}(\hat\theta_1,\cdots,\hat\theta_{n-1})$. For the same reason, we have $\theta_{n-1} = \hat\theta_{n-1}, \cdots, \theta_2 = \hat\theta_2$.

Finally $(\sin\theta_1, \cos\theta_1) = (\sin\hat\theta_1, \cos\hat\theta_1)$. As $\theta_1, \hat\theta_1 \in (0, 2\pi)$, $\theta_1 = \hat\theta_1$. Thus φ_n is one to one

(c) ~~If $x \in S^n - \text{Image}\,\varphi_n$, then $\prod_{i=1}^{n} \sin\theta_i = 0$. This is because if $\prod_{i=1}^{n}\sin\theta_i \neq 0$, $\varphi(\theta_1\cdots\theta_n)$~~ It is obvious that $\hat\varphi_n : U' \to R^{n+1}$ with $U' = \{(\theta_1\cdots\theta_n) \in R^n \mid 0 \le \theta_1 < 2\pi, 0 \le \theta_i < \pi\ i\in[2,n]\}$ maps onto S^n. So if $x = \hat\varphi(\theta_1\cdots\theta_n) \in S^n - \text{Image}\,\varphi$, then $(\theta_1\cdots\theta_n) \in U' \setminus U$.

so $\prod_{i=1}^{n} \sin\theta_i = 0$ So $x_1 = 0$. Thus $S^n - \text{Image}\,\varphi$ is contained in the $(n-1)$-sphere $\{(x_1\cdots x_{n+1}) \in S^n \mid x_1 = 0\}$. So $V(\varphi_n) = V(S^n)$

(d) $|\xi_n| = \left|\begin{vmatrix} \sin\theta_n \cdot J_{n-1} & \varphi_{n-1}\cos\theta_n & \varphi_{n-1}\sin\theta_n \\ 0 & -\sin\theta_n & \cos\theta_n \end{vmatrix}\right| = \left|\begin{vmatrix} \sin\theta_n J_{n-1} & \varphi_{n-1}\cos\theta_n & \varphi_{n-1}\frac{1}{\sin\theta_n} \\ 0 & -\sin\theta_n & 0 \end{vmatrix}\right| = (\sin\theta_n)^n |J_{n-1}\ \varphi_{n-1}|$
$= (\sin\theta_n)^n |\xi_{n-1}|$

So $V(\varphi_n) = \int_0^\pi (\sin\theta_n)^n d\theta_n\, V(\varphi_{n-1})$ for $n \ge 3$, $V(\varphi_2) = 4\pi$.

(e) Note the fact: $I_n = \int_0^\pi (\sin\theta)^n d\theta = \frac{n-1}{n}\int_0^\pi (\sin\theta)^{n-2} d\theta = \frac{n-1}{n} I_{n-2}$ for $n \ge 2$.

$I_1 = 2$, $I_2 = \pi/2$, $I_0 = \pi$. $I_n = \frac{(n-1)!!}{n!!}\pi$ if n is even and $I_n = \frac{(n-1)!!}{n!!}2$ if n is odd.

So $V(\varphi_n) = I_n \cdots I_3 \cdot V(\varphi_2) = 4\prod_{k=1}^{n} I_k$. $(n \ge 2)$.

17.9 Denote $V_i = \frac{\partial\varphi}{\partial u_i}$ $(i=1,2)$ $N = V_1 \times V_2 / \|V_1 \times V_2\|$.

$A(\varphi) = \int_U \begin{vmatrix} V_1 \\ V_2 \\ V_1\times V_2 \end{vmatrix} / \|V_1\times V_2\| = \int_U (V_1\times V_2)\cdot(V_1\times V_2) / \|V_1\times V_2\| = \int_U \|V_1\times V_2\|$

17.10 (a) By Ex 14.9, W is normal vector field along φ. $\begin{vmatrix} E_1(p) \\ E_n(p) \\ W \end{vmatrix} = \sum_{i=1}^{n} W_i^2 \ge 0$. So $W/\|W\|$ is the orientation vector field along φ

(b) $V(\varphi) = \int_U \begin{vmatrix} E_1(p) \\ E_n(p) \\ W/\|W\| \end{vmatrix} = \int_U W\cdot W/\|W\| = \int_U \|W\|$

17.11 Let ~~$E = (e_1,\cdots,e_n)$, with~~ $e_i = (0,\cdots 0,1,0\cdots 0) \in R^n$, $A = (E_1^\varphi,\cdots E_n^\varphi, N^\varphi)$, $B = (E_1^{\varphi\circ h},\cdots, E_n^{\varphi\circ h}, N^{\varphi\circ h})$

As N^{φ} is orientation vector field along φ. So $|B|>0$.

As $\psi = \varphi\circ h$, $A = (J^{\varphi}\circ h\cdot J_h\cdot e_1, \cdots, J^{\varphi}\circ h\cdot J_h\cdot e_n, N^{\varphi}\circ h) = (J^{\varphi}\circ h\cdot J_h, N^{\varphi}\circ h)$ if we assume $N^{\psi}=N^{\varphi}\circ h$.

Then $B=(J^{\varphi}\circ h, N^{\varphi}\circ h)$, $A^TB=\begin{pmatrix} J_h^T(J^{\varphi}\circ h)^T(J^{\varphi}\circ h) & 0 \\ 0 & 1\end{pmatrix}$ $B^TB=\begin{pmatrix}(J^{\varphi}\circ h)^T(J^{\varphi}\circ h) & 0 \\ 0 & 1\end{pmatrix}$

The zero's are because: $(N^{\varphi}\circ h)^T\cdot(J^{\varphi}\circ h)e_i=0$ by definition of N^{φ} and $(N^{\varphi}\circ h)^T(J^{\varphi}\circ h)\cdot J_h e_i = 0$ by the fact that $\{e_1\cdots e_n\}$ forms a basis of R^n so $J_h e_i$ can be written as a linear combination of $\{e_1\cdots e_n\}$.

So $|A^TB| = |J_h|\cdot|(J^{\varphi}\circ h)^T(J^{\varphi}\circ h)|$. $|B^TB| = |(J^{\varphi}\circ h)^T(J^{\varphi}\circ h)|$

So $|A|\cdot|B| = |J_h|\cdot|B|^2$. As $|J_h|>0$. $|B|>0$ so $|A|=|J_h||B|>0$

So $N^{\psi}=N^{\varphi}\circ h$ satisfies all the conditions to be orientation vector field.

17.12 (a) First prove $w(V_1\cdots V_i\cdots V_j\cdots V_K) = w(V_1\cdots V_i\cdots V_j+\alpha V_i,\cdots V_K)$ where $\alpha\in R$.

This is because the latter $= w(V_1\cdots V_i\cdots V_j\cdots V_K) + \alpha\, w(V_1\cdots V_i,\cdots V_i,\cdots V_K)$ and $w(V_1\cdots V_i,\cdots V_i\cdots V_K)=0$ by skewsymmetry.

If $\{V_1\cdots V_K\}$ is linearly dependent, then exist $\alpha_1\cdots\alpha_n\in R$, s.t. $\sum_{i=1}^{K}\alpha_i V_i=0$ and $\sum_{i=1}^{n}\alpha_i^2\neq 0$. So ~~$w(V_1\cdots V_K) = w(V_1$~~ assume $\alpha_i\neq 0$, then

$w(V_1\cdots V_i\cdots V_K) = \frac{1}{\alpha_i}w(V_1\cdots \alpha_i V_i,\cdots V_K) = \frac{1}{\alpha_i}w(V_1\cdots,\alpha_i V_i+\alpha_1 V_1,\cdots V_K)$

$= \cdots = \frac{1}{\alpha_i}w(V_1,\cdots,\sum_{i=1}^{K}\alpha_i V_i, V_K) = 0.$

(b) If $K>n$, then $\{V_1\cdots V_K\}$ must be linearly dependent, so $w=0$.

17.13 (a) $\xi(V_1\cdots V_n)^2 = \left|\begin{pmatrix}V_1^T\\ \vdots \\ V_n^T \\ N\end{pmatrix}(V_1\cdots V_n\, N)\right| = \begin{vmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{vmatrix} = 1$ So $\xi(V_1\cdots V_n)=\pm 1$

and $\xi(V_1\cdots V_n)=1$ iff $\{V_1\cdots V_n\}$ is consistent with N.

(b) We only need to prove $w(U_1,\cdots,U_n) = w(V_1\cdots V_n)\cdot\xi(U_1\cdots U_n)$ for any $\{U_1\cdots U_n\}\in S_p$, and $V_1\cdots V_n$ is arbitrary orthonormal basis for S_p consistent with the orientation N on S. As $\{V_1\cdots V_n\}$ forms a basis of S_p, so there exist $\alpha_{ij}\in R$ s.t. $U_i=\sum_{j=1}^{n}\alpha_{ij}V_j$

So $w(U_1\cdots U_n) = \sum_{i_1\cdots i_n}\alpha_{1i_1}\cdots\alpha_{ni_n}w(V_{i_1},\cdots,V_{i_n})$. If $i_p=i_q$ $(p\neq q)$ then $w(V_{i_1}\cdots V_{i_n})=0$

So $w(U_1\cdots U_n) = \sum_{\sigma}\alpha_{1\sigma(1)}\cdots\alpha_{n\sigma(n)}w(V_{\sigma(1)},\cdots,V_{\sigma(n)})$ ~~(sign σ)~~. (1)

Likewise $\xi(U_1\cdots U_n) = \sum_{\sigma}\alpha_{1\sigma(1)}\cdots\alpha_{n\sigma(n)}\xi(V_{\sigma(1)},\cdots,V_{\sigma(n)})$ ~~(sign σ)~~ (2)

by question (a)

Notice $w(V_{\sigma(1)}\cdots V_{\sigma(n)}) = (\text{sign}\,\sigma)\,w(V_1\cdots V_n)^{(3)}$, $\xi(V_{\sigma(1)}\cdots V_{\sigma(n)}) = (\text{sign}\,\sigma)\,\xi(V_1\cdots V_n) = \text{sign}\,\sigma^{(4)}$

So continuing (1) and plugging (3)(4) into (1) we have $w(U_1\cdots U_n) = \sum_{\sigma}\alpha_{1\sigma(1)}\cdots\alpha_{n\sigma(n)}\left(\xi(V_{\sigma(1)},\cdots,V_{\sigma(n)})\,w(V_1\cdots V_n)\right) = w(V_1\cdots V_n)\cdot\sum_{\sigma}\alpha_{1\sigma(1)}\cdots\alpha_{n\sigma(n)}\xi(V_{\sigma(1)}\cdots V_{\sigma(n)})$

$\overset{\text{by}(2)}{=} w(V_1\cdots V_n)\,\xi(U_1\cdots U_n)$

17.14 (a) ~~linear~~ multilinearity is obvious. We only need to prove skewsymmetry. To this end, we only need to prove for $\forall i,j \in \{1, \cdots k+l\}$, $(w_1 \wedge w_2)(v_1 \cdots v_i \cdots v_j \cdots v_{k+l})$ $= -(w_1 \wedge w_2)(v_1 \cdots v_j \cdots v_i \cdots v_{k+l})$.

For $\forall \sigma$, ~~if~~ let p, q s.t. $\sigma(p) = i, \sigma(q) = j$. If $p, q \leq k$, then v_i, v_j both appear in w_1 under such σ, so swapping $v_i\ v_j$ will just inverse the sign. The same happens if $p, q > k$.

If $p \leq k, q > k$, then look at $\hat\sigma$ which is the same as σ except $\hat\sigma(p) = j$, $\hat\sigma(q) = i$.

So $\operatorname{sign}\hat\sigma = -\operatorname{sign}\sigma$. For $(w_1 \wedge w_2)(\underset{\operatorname{sign}\hat\sigma}{v_1} \cdots v_i \cdots v_j \cdots v_{k+l})$ we have summands

$(\operatorname{sign}\sigma) w_1(\cdots v_i \cdots) w_2(\cdots v_j \cdots) - (\operatorname{sign}\sigma) w_1(\cdots v_j \cdots) w_2(\cdots v_i \cdots)$.

For $(w_1 \wedge w_2)(v_1 \cdots v_j \cdots v_i \cdots v_{k+l})$, we have summands

$(\operatorname{sign}\sigma) w_1(\cdots v_j \cdots) w_2(\cdots v_i \cdots) - (\operatorname{sign}\sigma) w_1(\cdots v_i \cdots) w_2(\cdots v_j \cdots)$.

So ~~for~~ the summands for swapped $v_i\ v_j$ ~~are~~ have opposite sign.

This also happens to $p > k, q \leq k$. So in all $(w_1 \wedge w_2)(v_1 \cdots v_i \cdots v_j \cdots v_{k+l}) = -(w_1 \wedge w_2)(v_1 \cdots v_j \cdots v_i \cdots v_n)$.

(b) We only need to prove that if

$(\sigma(1) \cdots \sigma(k), \sigma(k+1), \cdots \sigma(k+l)) = (\hat\sigma(l+1) \cdots \hat\sigma(k+l), \hat\sigma(1), \cdots, \sigma(l))$, i.e.

$w_1(v_{\sigma(1)} \cdots v_{\sigma(k)}) \cdot w_2(v_{\sigma(k+1)} \cdots v_{\sigma(k+l)}) = w_2(v_{\sigma(1)} \cdots v_{\sigma(l)}) \cdot w_1(v_{\sigma(l+1)} \cdots v_{\sigma(l+k)})$, then

$\operatorname{sign}\sigma = (-1)^{kl} \operatorname{sign}\hat\sigma$. This boils down to how many number of swaps is needed in order to change $(a_1 \cdots a_k a_{k+1} \cdots a_{k+l})$ to $(a_{k+1} \cdots a_{k+l} a_1 \cdots a_k)$, and we only care about the odd/even of the number. One schedule is pushing a_{k+1} ahead ~~for~~ by swapping with the element to its left for k times, i.e. $(a_1 \cdots a_{k-1} a_k a_{k+1}) \to (a_1 \cdots a_{k-1} a_{k+1} a_k) \to (a_1 \cdots a_{k+1} a_{k-1} a_k) \to \cdots \to (a_{k+1} a_1 \cdots a_k \cdots)$. Doing the same for $a_{k+2}, \cdots a_{k+l}$, then we change ~~$(a_1 \cdots a_{k+l})$~~ $(a_1 \cdots a_k a_{k+1} \cdots a_{k+l})$ to $(a_{k+1} \cdots a_{k+l} a_1 \cdots a_k)$ in kl steps.

Since the odd/even of step number is independent of schedule, we proved $\operatorname{sign}\sigma = (-1)^{kl} \operatorname{sign}\hat\sigma$.

(c) $w_1 \wedge (w_2 + w_3) = \frac{1}{k!l!} \sum_\sigma (\operatorname{sign}\sigma) w_1(v_{\sigma(1)} \cdots v_{\sigma(k)})(w_2 + w_3)(v_{\sigma(k+1)} \cdots v_{\sigma(k+l)})$

$= \frac{1}{k!l!} \sum_\sigma (\operatorname{sign}\sigma) w_1(v_{\sigma(1)} \cdots v_{\sigma(k)}) w_2(v_{\sigma(k+1)} \cdots v_{\sigma(k+l)}) + \frac{1}{k!l!} \sum_\sigma (\operatorname{sign}\sigma) w_1(v_{\sigma(1)} \cdots v_{\sigma(k)}) w_3(v_{\sigma(k+1)} \cdots v_{\sigma(k+l)})$

$= (w_1 \wedge w_2) + (w_1 \wedge w_3)$.

(d) $(w_1 \wedge w_2) \wedge w_3 = \frac{1}{k!l!m!(k+l)!} \sum_{\sigma, \hat\sigma} (\operatorname{sign}\sigma)(\operatorname{sign}\hat\sigma) w_1(v_{\sigma(\hat\sigma(1))}, \cdots v_{\sigma(\hat\sigma(k))}) w_2(v_{\sigma(\hat\sigma(k+1))} \cdots v_{\sigma(\hat\sigma(k+l))}) w_3(v_{\sigma(k+l+1)} \cdots v_{\sigma(k+l+m)})$ (*)

where σ is a permutation of $1 \cdots (k+l+m)$ and $\hat\sigma$ is a permutation of $1 \cdots k+l$.

Notice $(\operatorname{sign}\sigma) \cdot (\operatorname{sign}\hat\sigma) = \operatorname{sign}(\sigma \circ \hat\sigma)$. (we can define $\hat\sigma(i) = i$ for $i > k+l$).

For each $w_1(v_{i_1} \cdots v_{i_k}) w_2(v_{i_{k+1}} \cdots v_{i_{k+l}}) w_3(v_{i_{k+l+1}} \cdots v_{i_{k+l+m}})$, there exist $(k+l)!$ different combinations of σ and $\hat\sigma$ which finally results in this order of subscript by permutating from $(1, \cdots, k+l+m)$. In fact, for any $\hat\sigma$, there exists a unique σ, such that $\sigma \circ \hat\sigma$ yields above subscripts. Besides, all such combinations have the same sign of $\sigma \circ \hat\sigma$. So (*) is equal to $\frac{1}{k!\,l!\,m!}\sum_{\sigma} w_1(V_{\sigma(1)}\cdots V_{\sigma(k)})\,w_2(V_{\sigma(k+1)}\cdots V_{\sigma(k+l)})\,w_3(V_{\sigma(k+l+1)}\cdots V_{\sigma(k+l+m)})$ (1)

For the same reason, $w_1\wedge(w_2\wedge w_3)$ is also equal to (1).

Thus $(w_1\wedge w_2)\wedge w_3 = w_1\wedge(w_2\wedge w_3)$.

(e) First prove for $\forall k\in[1,n]$ $(w_1\wedge\cdots\wedge w_k)(X_1\cdots X_k)=1$. by induction

If $k=1$, then $w_1(X_1)=X_1(p)\cdot X_1(p)=1$. If it's true for k, then

$(w_1\wedge\cdots\wedge w_{k+1})(X_1\cdots X_{k+1})=\frac{1}{k!}\sum_{\sigma}(\text{sign}\,\sigma)(w_1\wedge\cdots\wedge w_k)(X_{\sigma(1)}\cdots X_{\sigma(k)})\,w_{k+1}(X_{\sigma(k+1)})$

If $\sigma(k+1)\neq k+1$, then $w_{k+1}(X_{\sigma(k+1)})=X_{k+1}\cdot X_{\sigma(k+1)}=0$. So we

only look at those σ, s.t. $\sigma(k+1)=k+1$. so $\sigma(1)\cdots\sigma(k)$ is a permutation of $1,2,\cdots,k$ ~~Let $\tilde\sigma(i)=\sigma(i)$ $i=1\cdots k$~~, Then

$(w_1\wedge\cdots\wedge w_{k+1})(X_1\cdots X_{k+1})=\frac{1}{k!}\sum_{\sigma}(\text{sign}\,\sigma)^2(w_1\wedge\cdots\wedge w_k)(X_1\cdots X_k)$. as $w_1\wedge\cdots\wedge w_k$ is k-form

$=\frac{1}{k!}\cdot k!=1$. (Implicitly using the fact that when a k+1 permutation σ satisfies $\sigma(k+1)=k+1$, then its sign is equal to the k permutation $\tilde\sigma$ defined as $\tilde\sigma(i)=\sigma(i)$ $i=1\cdots k$.)

So $(w_1\wedge\cdots\wedge w_k)(X_1\cdots X_k)=1$ for all $k=1\cdots n$.

This step is not necessary. We just follow the hint on textbook.

Next prove for $\forall k\in[1,n]$, $i>k$, $X_i\lrcorner(w_1\wedge\cdots\wedge w_k)=0$. Actually $\forall V_1\cdots V_{k-1}\in S_p$,
$X_i\lrcorner(w_1\cdots w_k)(V_1\cdots V_{k-1})=(w_1\wedge\cdots\wedge w_k)(X_i(p),V_1,\cdots,V_{k-1})$. Expanding as in the definition,
if $X_i(p)$ appear in $w_k(\cdot)$, then $w_k(X_i(p))=X_k(p)\cdot X_i(p)=0$
If $X_i(p)$ appear in $w_1\wedge\cdots\wedge w_{k-1}$, then by some induction like proof, it's easy
to show $(w_1\wedge\cdots\wedge w_k)(\cdots,X_i(p),\cdots)=0$. So $X_i\lrcorner(w_1\cdots w_k)=0$. $i>k$, $k\in[1,n]$

Finally, as $\{X_1\cdots X_{n+1}\}$ is an orthonormal basis for S_p, by Ex 17.13, $f(p)=1$

because $(w_1\wedge\cdots\wedge w_n)(X_1(p)\cdots X_n(p))=1$. So $w_1\wedge\cdots\wedge w_n=\xi$

17.15 (a) multilinearity is obvious. $f^*w(V_{\sigma(1)},\cdots,V_{\sigma(k)})=w(df(V_{\sigma(1)}),\cdots,df(V_{\sigma(k)}))$

$=(\text{sign}\,\sigma)\,w(df(V_1),\cdots,df(V_k))=(\text{sign}\,\sigma)f^*w(V_1,\cdots,V_k)$

As w, df are smooth. f^*w is also smooth.

(b) $\int_\varphi f^*w=\int_U w(df(E_1^\varphi),\cdots,df(E_k^\varphi))=\int_U w(E_1^{f\circ\varphi},\cdots E_k^{f\circ\varphi})=\int_{f\circ\varphi} w$

(c) Suppose $\{f_i\}$ is a partition of unity on S subordinate to a collection $\{\varphi_i\}$ of one to one local parametrizations of S. We prove first that $\{f_i\circ f^{-1}\}$ is a partition of unity of $\tilde S$, subordinate to a collection $\{f\circ\varphi_i\}$ of one to one local parametrizations of $\tilde S$.

① $\forall q\in\tilde S$, $f^{-1}(q)\in S$ (as f is diffeomorphism), So $f_i(f^{-1}(q))\geq 0$ $i=1\cdots m$

② $\forall q\in\tilde S$, $f^{-1}(q)\in S$, thus $\sum_{i=1}^m f_i(f^{-1}(q))=1$

③ as f,φ_i are both one to one, so $f\circ\varphi_i$ is one to one. As φ_i is regular and m=n and f ~~is dif~~ is diffeomorphism, so Jf is invertible and $f\circ\varphi_i$ is regular. so $f\circ\varphi_i$ is ~~also~~ local parametrization of $\tilde S$. Besides, f is orientation preserving and $f\circ\varphi_i$ must be open as f^{-1} is smooth ~~and invertible~~

Suppose f_i is identically zero outside the image under φ_i of a compact subset

B_i of U_i. ~~then as f is smooth. $f(B_i)$ is also compact. $\forall x\in f(B_i)$,~~ and $x\in$
let $x\notin f(\varphi_i(B_i))$. $(x\in f(\varphi_i(U_i)))$, then if $f_i(f^{-1}(x))\neq 0$, then $f^{-1}(x)\in\varphi_i(B_i)$, then $x=f(f^{-1}(x))\in f(\varphi_i(B_i))$, which contradicts with our assumption. so $f_i(f^{-1}(x))=0$
So $f_i\circ f^{-1}$ is identically 0 outside the image under $f\circ\varphi_i$ of a compact subset B_i of U_i.
Combining ①-③, we conclude $\{f_i\circ f^{-1}\}$ is a partition of unity of $\tilde{S}$, subordinate to a collection $\{f\circ\varphi_i\}$ of one-to-one local parametrization of $\tilde{S}$.
Finally. $\int_S f^*w=\sum_i\int_{\varphi_i}f_i f^*w=\sum_i\int_{U_i}f_i\circ\varphi_i\cdot w(df(E_1^{\varphi_i}),\dots,df(E_k^{\varphi_i}))$
$=\sum_i\int_{U_i}f_i\circ f^{-1}\circ f\circ\varphi_i\cdot w(E_1^{f\circ\varphi_i},\dots,E_k^{f\circ\varphi_i})=\int_{f_i\circ\varphi_i}f_i\circ f^{-1}w=\int_{\tilde S}W$

~~17.16~~

17.16 For $\forall p\in S^n$. If $v_1\cdots v_n\in S_p$ is a basis of S_p and $\begin{vmatrix}v_1\\ \vdots\\ v_n\\ N(p)\end{vmatrix}>0$. then $df(v_i)=-v_i$, $N(f(p))=-N(p)$, so $\begin{vmatrix}df(v_1)\\ \vdots\\ df(v_n)\\ N(f(p))\end{vmatrix}=(-1)^{n+1}\begin{vmatrix}v_1\\ \vdots\\ v_n\\ N(p)\end{vmatrix}$, which is positive iff n is odd.

17.17(a) $\lim_{t\to0^+}h(t)=0$, $h'(t)=\frac{1}{t^2}e^{-\frac1t}$, so $\lim_{t\to0^+}h'(t)=0$. Generally, $h^{(n)}(t)$ must be in the form of $h^{(n)}(t)=P(\frac1t)\cdot e^{-\frac1t}$, where $P(x)$ is a polynomial function of x, with finite degree. ~~Actually,~~
~~$h''(t)$~~ So $\lim_{t\to0^+}h^{(n)}(t)=0$. Obviously $\lim_{t\to0^-}h^{(n)}(t)=0$ So h is smooth

(b) $h_r(t)=h(u(t))$, where $u(t)=r^2-t^2$. Since both $u(t)$, $h(u)$ are smooth, so $h_r(t)$ is smooth.
In the proof of Thm 4. φ_p^{-1} is smooth, $u(q)=-\|q-\varphi_p^{-1}(p)\|^2+r_p^2$ is also smooth wrt $q\in\mathbb{R}^{n+1}$
So $g_p(q)=h(u(\varphi_p^{-1}(q)))$ is smooth.

$$\begin{tikzcd}
s \quad \underset{}{\overset{\varphi^{-1}(W)}{U}} \arrow[dr, "\varphi"] & P \\
 & S\supseteq W \\
t \quad \underset{\psi^{-1}(W)}{V} \arrow[uu, "h"] \arrow[ur, "\psi"'] &
\end{tikzcd}$$

17.18 Since φ,ψ are both one to one local parametrization
So $\varphi|_{\varphi^{-1}(W)}$ and $\psi|_{\psi^{-1}(W)}$ are both bijective ~~to W~~ from $\varphi^{-1}(W)$ or $\psi^{-1}(W)$ to W. So φ^{-1} and ψ^{-1} are bijective,
thus $\varphi^{-1}\circ\psi|_{\psi^{-1}(W)}$ is also bijective. φ and ψ are both smooth and regular, so φ^{-1},ψ^{-1} must be smooth. So $\varphi^{-1}\circ\psi|_{\psi^{-1}(W)}$ is smooth, and its inverse $\psi^{-1}\circ\varphi|_{\varphi^{-1}(W)}$ is also smooth. So $\varphi^{-1}\circ\psi|_{\psi^{-1}(W)}$ is diffeomorphism.
The textbook only defines "orientation preserving" for a map between two oriented n-surfaces in $\mathbb{R}^{n+1}$ at a regular point, so we don't know what it means by h being orientation preserving, because h maps from $\psi^{-1}(W)$ (open set) to $\varphi^{-1}(W)$ (open set). However we can still prove that $|J_h|>0$, thus $\psi|_{\psi^{-1}(W)}$ is reparametrization of $\varphi|_{\varphi^{-1}(W)}$.
For any point $p\in W$, suppose $s=\varphi^{-1}(p)$, $t=\psi^{-1}(p)$. Since both φ and ψ are local parametrizations of S, we have $s=\varphi^{-1}(\psi(t))=h(t)$. and

$$A = \begin{pmatrix} (J_\varphi(s)\cdot e_1)^T \\ \vdots \\ (J_\varphi(s)\, e_n)^T \\ N^T(p) \end{pmatrix},\ |A|>0,\quad B = \begin{pmatrix} (J_\psi(t)\, e_1)^T \\ \vdots \\ (J_\psi(t)\, e_n)^T \\ N^T(p) \end{pmatrix},\ |B|>0. \quad \text{But } J_\psi(t) = J_{\varphi\circ h}(t)\cdot J_h(t) = J_\varphi(s) J_h(t)$$

as $\psi = \varphi \circ h$.

where $N(p)$ is the orientation of S.

$$AB^T = \begin{pmatrix} J_\varphi^T(s) \\ N^T(p) \end{pmatrix} \left(J_\varphi(s) J_h(t),\ N(p) \right) = \begin{pmatrix} J_\varphi^T(s)\cdot J_\varphi(s)\cdot J_h(t) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{So } |A|\cdot|B| = |J_\varphi^T(s) J_\varphi(s)|\cdot|J_h(t)|$$

As $J_\varphi^T J_\varphi$ is positive semi-definite, $|J_\varphi^T(s) J_\varphi(s)| \geq 0$, But $|A|,|B|>0$. So $\left|\overset{J_h(t)}{\cancel{\ \ }}\right| > 0$.

Since p is any point on W and ψ is bijective, so $|J_h(t)|>0$ for any $t \in \psi^{-1}(W)$

Thus $\psi|_{\psi^{-1}(W)} = \varphi\circ h|_{\psi^{-1}(W)}$ is reparametrization of $\varphi|_{\varphi^{-1}(W)}$

17.19 Denote $x = X(p) = (x_1, x_2, x_3)$, $y = Y(p) = (y_1, y_2, y_3)$

$$(W_X \wedge W_Y)(v,w) = W_X(v)W_Y(w) - W_X(w)W_Y(v) = (x\cdot v)\cdot(y\cdot w) - (x\cdot w)(y\cdot v)$$

$$= \left(\sum_{i=1}^3 x_i v_i\right)\left(\sum_{i=1}^3 y_i w_i\right) - \left(\sum_{i=1}^3 x_i w_i\right)\left(\sum_{i=1}^3 y_i v_i\right)$$

$$(X\times Y)(p)\cdot(v\times w) = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)\cdot(v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1)$$

$$= x_2v_2y_3w_3 + x_3v_3y_2w_2 + x_3v_3y_1w_1 + x_1v_1y_3w_3 + x_1v_1y_2w_2 + x_2v_2y_1w_1$$

$$- x_2y_3v_3w_2 - x_3y_2v_2w_3 - x_3y_1v_1w_3 - x_1y_3v_3w_1 - x_1y_2v_2w_1 - x_2y_1v_1w_2$$

$$= \left(\sum_{i=1}^3 x_i v_i\right)\left(\sum_{i=1}^3 y_i w_i\right) - \left(\sum_{i=1}^3 x_i w_i\right)\left(\sum_{i=1}^3 y_i v_i\right)$$

So $(W_X \wedge W_Y)(v,w) = (X\times Y)(p)\cdot(v\times w)$