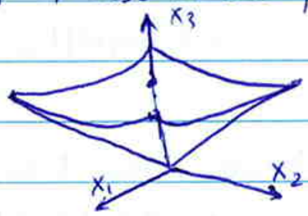
3.1 n=1 $f=x_1^2-x_2^2$ $f^{-1}(-1)$ is $x_1^2=x_2^2+1$ $\nabla f=(2x_1,-2x_2)$ so $\nabla f\neq 0$, no such p
$f^{-1}(1)$ also doesn't have such p. $f^{-1}(0)$, $\nabla f(0,0)=(0,0)$, $f^{-1}(0)$ is $x_1=\pm x_2$
Its tangent space is $\{\lambda(1,1),\lambda(1,-1)|\lambda\in R\}\neq[\nabla f(0,0)]^\perp=R^2$
n=2 $f=x_1^2+x_2^2-x_3^2$ $f^{-1}(-1)$ is $x_1^2+x_2^2=x_3^2+1$. $\nabla f\neq 0$ no such p. $f^{-1}(1)$ also no such p
$f^{-1}(0)$: $x_3^2=x_1^2+x_2^2$ at p=(0,0), the tangent space
at (0,0,0) is all vectors $V=(x_1,x_2,x_3)$ where V is 45° to x_3 axis
$|V\cdot(0,0,1)/\|V\||=\frac{\sqrt{2}}{2}$ i.e. $x_3^2=x_1^2+x_2^2\neq[\nabla f(0)]^\perp$



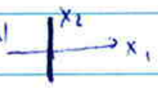
3.2 (a) the example in 3.1 with n=1, c=0. $f^{-1}(0)$: $x_1=\pm x_2$ (1,1), (1,-1) ∈ S, (1,0) ∉ S
(b) $f(x_1\cdots x_{n+1})=c$. $S=f^{-1}(c)$, tangent space $=R_p^{n+1}$

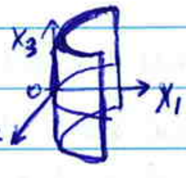
3.4 $f\circ\alpha=c \Leftrightarrow \frac{d(f(\alpha(t)))}{dt}=0 \Leftrightarrow \nabla f(\alpha(t))\cdot\dot\alpha(t)=0 \Leftrightarrow \dot\alpha\perp\nabla f(\alpha)$ ∀t.

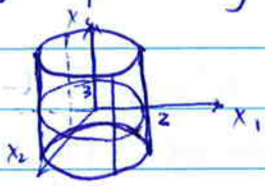
3.5 α is integral curve of ∇f ⇒ $\dot\alpha=\nabla f(\alpha)$
(a) $\frac{d}{dt}f(\alpha(t))=\nabla f(\alpha(t))\cdot\dot\alpha(t)=\|\nabla f(\alpha(t))\|^2$
(b) $\frac{d}{dt}f(\beta(s_0))=\nabla f(\beta(s_0))\cdot\dot\beta(s_0)=\nabla f(\alpha(t_0))\cdot\dot\beta(s_0)$. As $\|\dot\beta(s_0)\|=\|\dot\alpha(t_0)\|$
it is maximized when $\dot\beta(s_0)=\dot\alpha(t_0)=\nabla f(\alpha(t_0))$, then
$\frac{d}{dt}f(\beta(s_0))=\|\nabla f(\alpha(t_0))\|^2=\frac{d}{dt}f(\alpha(t_0))$ by (a)

4.3 Consider $S=f^{-1}(c)$. ∀p∈S, p is an extreme point of g on S.
By Lagrange Theorem, $\nabla g(p)=\lambda\cdot\nabla f(p)$ ∀p∈S. $\lambda\neq 0$ because $\nabla g(p)\neq 0$ for all p∈S

4.4 See http://users.rsise.anu.edu.au/~xzhang/dg_thorpe/monkey.jpg

4.5 (a) 

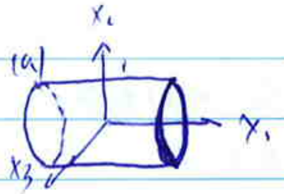
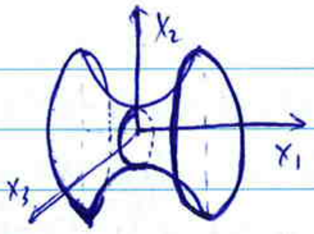
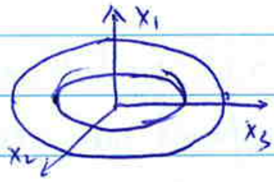
the x_2-axis (b) 

(c) 

ellipse on $x_3=0$

4.6 

4.7 $g(x_1,x_2,x_3)=f(x_1,(x_2^2+x_3^2)^{1/2})$, then $\frac{\partial g}{\partial x_1}=\frac{\partial f}{\partial x_1}$. Denote $u=(x_2^2+x_3^2)^{1/2}$
then $\frac{\partial g}{\partial x_2}=\frac{\partial f}{\partial u}x_2(x_2^2+x_3^2)^{-1/2}$, $\frac{\partial g}{\partial x_3}=\frac{\partial f}{\partial u}x_3(x_2^2+x_3^2)^{-1/2}$. If $\nabla g(p)=0$, then
$\frac{\partial g}{\partial x_2}=\frac{\partial g}{\partial x_3}=0$, i.e. $0=(\frac{\partial g}{\partial x_2})^2+(\frac{\partial g}{\partial x_3})^2=[\frac{\partial f}{\partial u}]^2=0$. So $\frac{\partial f}{\partial u}=0$. Besides $\frac{\partial f}{\partial x_1}=\frac{\partial g}{\partial x_1}=0$
So $\nabla f=0$ at p, which contradicts with the fact that ∇f ~~is a surface~~ $\neq 0$

4.8 (a)  (b)  (c)  ref Fig 14.5 Pg. 112 (change axis)

4.9 $f(x) = x_3^2 + x_4^2 - 1$ $\quad S = f^{-1}(0)$. $\nabla f = (0, 0, 2x_3, 2x_4)$ $\quad \nabla f = 0 \Rightarrow x_3 = x_4 = 0 \Rightarrow$ not on S

4.10 $g(x_1, x_2, x_3, x_4) = f(x_1, x_2, (x_3^2 + x_4^2)^{1/2})$

4.11 By Lagrange Thm, $\nabla g = \lambda \nabla f$. $\Rightarrow \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = \lambda \begin{pmatrix} 2ax_1 + 2bx_2 \\ 2bx_1 + 2cx_2 \end{pmatrix} \Rightarrow \lambda \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Since $ac - b^2 > 0 \Rightarrow \det \begin{pmatrix} a & b \\ b & c \end{pmatrix} \neq 0 \Rightarrow \lambda \neq 0 \Rightarrow \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

At that point $g = (x_1, x_2) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda (x_1, x_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda$. Note λ^{-1} is eigenvalue of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

4.12 $g = x^T A x$ $\quad \nabla g = 2Ax$ $\quad f = \sum_{i=1}^{n} x_i^2$ $\quad \nabla f = 2x$ $\quad \nabla g = \lambda \nabla f \Rightarrow Ax = \lambda x$

$g(x) = \lambda x^T x = \lambda$ the eigenvalue of A

4.13 By Lagrange Thm, $\lambda \nabla f(p) = \nabla g(p)$. As $\nabla g(p) \neq 0$ $\lambda \neq 0$ $\quad \forall v$: $v \cdot \nabla g(p) = 0 \Leftrightarrow v \cdot \nabla f(p) = 0$

So tangent space of g through p is equal to tangent space of f through p

4.14 Let $g = \|x - p_0\|^2$. $S = f^{-1}(c)$. Since p is an extreme point of g on S

$\nabla g(p) = \lambda \nabla f(p)$ But $\nabla g(p) = 2(p - p_0)$. So $(p, p - p_0) \perp S_p$.

4.15 $\nabla \det(X) = \frac{1}{\det(X)} (X^{-1})^T$ So $\nabla \det(X) = 0$ is impossible.

4.16 (a) $\nabla \det(X) = \frac{1}{1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So $\left\langle \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_F = 0 \Rightarrow x_1 + x_4 = 0$

(b) $\nabla \det(\,) = \frac{1}{1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ So $SL(2)_q = \{ (p, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) : a - b - c + 2d = 0 \}$.

4.17 (a) The proof in 4.15 is independent of dimension

(b) $\nabla \det(p) = I$. So $SL(3)_p = \{ (p, M) \mid M \in \mathbb{R}^{3\times 3},\ tr(M) = 0 \}$.

5.1 ~~Only need to prove every point is connected to origin (0,0,1) $\forall x$, define~~

$\forall x_1, x_2$, consider parametrized curve. $\alpha(t) = x_1 \cos t + u \sin t$ ~~where $u = \frac{e_1 - x\cos\theta}{\sin\theta}$~~

~~then $\alpha(0) = x_1$~~, where $u = \frac{x_2 - x_1 \cos\theta}{\sin\theta}$, here $\theta = \cos^{-1}(x_1 \cdot x_2)$ if $\sin\theta \neq 0$ ~~here $\theta = \arccos(x$ (if $\sin\theta \neq 0$)~~

then $\alpha(0) = x_1$ $\quad \alpha(\theta) = x_1 \cos\theta + \frac{x_2 - x_1 \cos\theta}{\sin\theta} \sin\theta = x_2$,

$\|u\| = \frac{1}{\sin^2\theta} (1 + \cos^2\theta - 2\cos\theta \cdot \langle x_1; x_2 \rangle) = \frac{1}{\sin^2\theta} (1 + \cos^2\theta - 2\cos^2\theta) = 1$, $\langle u; x_1 \rangle = \frac{\langle x_1, x_2 \rangle - \cos\theta}{\sin\theta} = \frac{\cos\theta - \cos\theta}{\sin\theta} = 0$