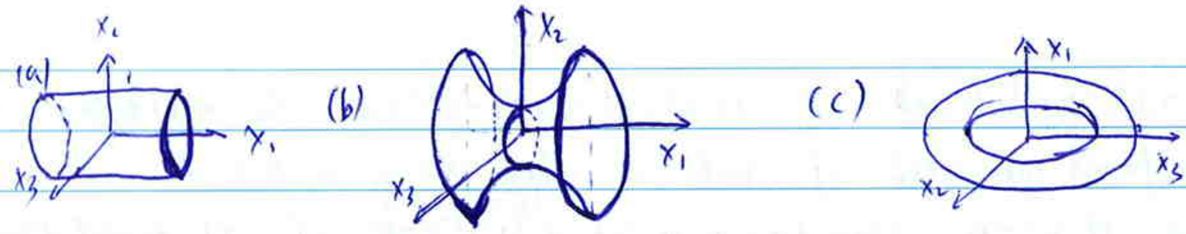
4.8 (a) (b) (c)



ref Fig 14.5 Pg. 112 (change axis)

4.9 $f(x) = x_3^2 + x_4^2 - 1$ $S = f^{-1}(0)$. $\nabla f = (0, 0, 2x_3, 2x_4)$ If $\nabla f = 0 \Rightarrow x_3 = x_4 = 0 \Rightarrow$ not on S

4.10 $g(x_1, x_2, x_3, x_4) = f(x_1, x_2, (x_3^2 + x_4^2)^{1/2})$

4.11 By Lagrange Thm, $\nabla g = \lambda \nabla f$. $\Rightarrow \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = \lambda \begin{pmatrix} 2ax_1 + 2bx_2 \\ 2bx_1 + 2cx_2 \end{pmatrix} \Rightarrow \lambda \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Since $ac - b^2 > 0 \Rightarrow \det\begin{pmatrix} a & b \\ b & c \end{pmatrix} \neq 0 \Rightarrow \lambda \neq 0 \Rightarrow \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

At that point $g = (x_1, x_2)^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda (x_1, x_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda$. Note λ^{-1} is eigenvalue of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

4.12 $g = x^T A x$ $\nabla g = 2Ax$ $f = \sum_{i=1}^{n} x_i^2$. $\nabla f = 2x$ $\nabla g = \lambda \nabla f \Rightarrow Ax = \lambda x$

$g(x) = \lambda x^T x = \lambda$ the eigenvalue of A

4.13 By Lagrange Thm, $\lambda \nabla f(P) = \nabla g(P)$. As $\nabla g(P) \neq 0$ $\lambda \neq 0$ $\forall v$: $v \cdot \nabla g(P) = 0 \Leftrightarrow v \cdot \nabla f(P) = 0$

So tangent space of g through P is equal to tangent space of f through P

4.14 Let $g = \|x - P_0\|^2$. $S = f^{-1}(c)$. Since P is an extreme point of g on S

$\nabla g(P) = \lambda \nabla f(P)$ But $\nabla g(P) = 2(P - P_0)$. So $(P, P - P_0) \perp S_P$.

4.15 $\nabla \det(X) = \frac{1}{\det(X)} (X^{-1})^T$ So $\nabla \det(X) = 0$ is impossible.

4.16 (a) $\nabla \det(X) = \frac{1}{1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So $\left\langle \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_F = 0 \Rightarrow x_1 + x_4 = 0$

(b) $\nabla \det(g) = \frac{1}{1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ So $SL(2)_g = \{ (P, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) : a - b - c + 2d = 0 \}$.

4.17 (a) The proof in 4.15 is independent of dimension

(b) $\nabla \det(P) = I$. So $SL(3)_P = \{ (P, M) \mid M \in R^{3\times3}, tr(M) = 0 \}$

5.1 ~~Only need to prove every point is connected to origin (0,0,1) $\forall x$, define~~

$\forall x_1, x_2$, consider parametrized curve $\alpha(t) = x_1 \cos t + u \sin t$ ~~where $u = \frac{e_1 - x_1 \cos\theta}{\sin\theta}$~~

~~then $\alpha(0) = x_1$~~, where $u = \frac{x_2 - x_1 \cos\theta}{\sin\theta}$, here $\theta = \cos^{-1}(x_1 \cdot x_2)$ if $\sin\theta \neq 0$ ~~here $\theta = \arccos(x$ (if $\sin\theta \neq 0)$~~

then $\alpha(0) = x_1$ $\alpha(\theta) = x_1 \cos\theta + \frac{x_2 - x_1 \cos\theta}{\sin\theta} \sin\theta = x_2$,

$\|u\| = \frac{1}{\sin^2\theta} (1 + \cos^2\theta - 2\cos\theta \cdot \langle x_1, x_2 \rangle) = \frac{1}{\sin^2\theta} (1 + \cos^2\theta - 2\cos^2\theta) = 1$, $\langle u, x_1 \rangle = \frac{\langle x_1, x_2 \rangle - \cos\theta}{\sin\theta} = \frac{\cos\theta - \cos\theta}{\sin\theta} = 0$

So $\|\alpha(t)\| = \|x_1\|^2\cos t + \|u\|^2\sin t + \langle x_1, u\rangle \sin t \cos t = \cos^2 t + \sin^2 t = 1$ So $\alpha(t) \in S$.

So far, we've found the curve. If $\sin\theta = 0$. Then $x_1 = x_2$ or $x_1 = -x_2$

If $x_1 = x_2$, done. If $x_1 = -x_2$. then find a u, s.t. $\|u\| = 1$ and $\langle x_1, u\rangle = 0$ and $\alpha(t) = x_1 \cos t + u \sin t$

$\|\alpha(t)\| = 1$, $\alpha(0) = x_1$, $\alpha(\pi) = -x_1 = x_2$ □ QED.

Note. A easier way is by using ~~polo~~ polar angular axis.

Let $x_1 = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \cdots, \sin\theta_1\cdots\sin\theta_{n-1}\cos\theta_n, \sin\theta_1\cdots\sin\theta_n)$

$x_2 = (\cos\theta_1', \sin\theta_1'\cos\theta_2', \sin\theta_1'\sin\theta_2'\cos\theta_3', \cdots, \sin\theta_1'\cdots\sin\theta_{n-1}'\cos\theta_n', \sin\theta_1'\cdots\sin\theta_n')$

So we just need to find a continuous curve from $(\theta_1, \cdots, \theta_n) \to (\theta_1', \cdots \theta_n')$ in $[0, 2\pi)^n$

But $[0, 2\pi)^n$ is convex set, so just easily find $\beta(t)$. s.t. $\beta(t) \in [0, 2\pi)^n$

$\beta(0) = (\theta_1, \cdots, \theta_n)$ $\beta(t_0) = (\theta_1', \cdots, \theta_n')$. Then define

$\alpha(t) = (\cos\beta_1(t), \sin\beta_1(t)\cos\beta_1(t), \cdots, \sin\beta_1(t)\cdots\sin\beta_{n-1}(t)\cos\beta_n(t), \sin\beta_1(t)\cdots\sin\beta_n(t))$

5.2 If there exists $p, q \in S$, s.t. $g(p) = 1$, $g(q) = -1$. then as S is connected, there exists a continuous map $\alpha: [a,b] \to S$, s.t $\alpha(a) = p$, $\alpha(b) = q$

As $g \circ \alpha$ is continous, $g(\alpha(a)) = 1$. $g(\alpha(b)) = -1$, So there exists $c \in (a,b)$, s.t. $g(\alpha(c)) = 0$

But by definition of α, $\alpha(c) \in S$ which contradicts with $g(x) = \pm 1$ for $\forall x \in S$.

5.3 1-surface: $f(x_1, x_2) = (x_1 - 1)(x_1 + 1)$

Define $g(x_1, x_2) = \begin{cases} -1 & x_1 \in (-3/2, -1/2) \\ 1 & x_1 \in (1/2, 3/2) \end{cases}$ So g is smooth on S, but g is not constant

5.4 $N_1(p)$ and $N_2(p)$ are both smooth. $\|\pm p/r\| = 1$, $\pm p/r \in S_p^\perp$ $\nabla(\sum_{i=1}^n x_i^2) = 2(x_1, \cdots, x_n)^T$

5.5 (a)

rotate counterclock-wise by $\pi/2$

(b) $R_\theta(v,0) = \cos\theta \cdot (v,0) + \sin\theta \cdot (0,0,1) \times (v,0) = (v', 0)$ where $v' = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} v$

which is counterclock-wise rotation with angle θ.

(c) $\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 > 0$ So right-handed

5.6 Let θ denote the angle measured counterclock-wise from $(p, 1, 0)$ to the orientation direction $N(p)$, so that $N(p) = (p, \cos\theta, \sin\theta)$ So the positive tangent direction is $(\cos(\theta - \frac{\pi}{2}), \sin(\theta - \frac{\pi}{2})) = (\sin\theta, -\cos\theta)$. v is tangent to C at p, So $v/\|v\| = \pm(\sin\theta, -\cos\theta)$

But if $v/\|v\| = -(\sin\theta, -\cos\theta)$ then $\det\begin{pmatrix} v/\|v\| \\ N(p) \end{pmatrix} = \det\begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} = -1$. which means inconsistent

So positive tangent $\Leftrightarrow$ consistent

5.7 (a)(b) just write out (c) take $u = (1,0,0), (0,1,0), (0,0,1)$ then get it

5.8 (a) consistent $\Leftrightarrow \det\begin{pmatrix} v \\ w \\ N(p)\end{pmatrix} > 0 \Leftrightarrow v\cdot(w \times N(p)) > 0 \Leftrightarrow N(p)\cdot(v \times w) > 0$

(b) Denote $\hat{x} \triangleq x/\|x\|$, consistent $\Leftrightarrow \hat{w}\cdot(N(p)\times\hat{v}) > 0$

~~[As $\{v,w\}$ is a basis of S_p, so there must exist θ. $\hat{w} = \cos\theta\,\hat{v} + \sin\theta\, N(p)\times\hat{v}$~~ As $\|\hat{w}\|=1$

(Proof) As $N(p)\cdot(N(p)\times\hat{v}) = \det\begin{pmatrix} N(p) \\ N(p) \\ \hat{v}\end{pmatrix} = 0$ So $N(p)\times\hat{v} \in S_p$.

$\hat{v}\cdot(N(p)\times\hat{v}) = \det\begin{pmatrix} \hat{v} \\ N(p) \\ \hat{v}\end{pmatrix} = 0$. So $\{N(p)\times\hat{v}, \hat{v}\}$ is an orthonormal basis of S_p

As $\|\hat{w}\| = 1$. So there exists θ s.t. $\hat{w} = \cos\theta\cdot\hat{v} + \sin\theta\cdot N(p)\times\hat{v}$

So $\hat{w}\cdot(N(p)\times\hat{v}) = \sin\theta$

So $\theta \in (0,\pi) \Leftrightarrow \hat{w}\cdot(N(p)\times\hat{v}) > 0 \Leftrightarrow \{v,w\}$ is consistent with N

5.9 (a) take $u = (1,0,0,0), (0,1,0,0) \cdots (0001)$ (b) just check

5.10 (a) $\det\begin{pmatrix} v_1 \\ \vdots \\ v_n \\ N\end{pmatrix} < 0 \Leftrightarrow \det\begin{pmatrix} v_1 \\ \vdots \\ v_n \\ -N\end{pmatrix} > 0$

(b) let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n\end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n\end{pmatrix}$ $\begin{pmatrix} W \\ N\end{pmatrix} = \begin{pmatrix} AV \\ N\end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} V \\ N\end{pmatrix}$ where $W = \begin{pmatrix} w_1 \\ \vdots \\ w_n\end{pmatrix}$, $V = \begin{pmatrix} v_1 \\ \vdots \\ v_n\end{pmatrix}$

So $\det\begin{pmatrix} W \\ N\end{pmatrix} = \det A \cdot \det\begin{pmatrix} V \\ N\end{pmatrix}$, thus consistency of W with N is identical to the consistency of V with N iff $\det A > 0$

6.1 ~~$N(S) = \{v \mid \|v\| \{v\in$~~ $n=1$ $N(S) = \{(0,1),(0,-1)\}$; $n=2$ $N(S) = \{(0,x_2,x_3) \mid x_2^2 + x_3^2 = 1\}$ $\in \mathbb{R}^3$

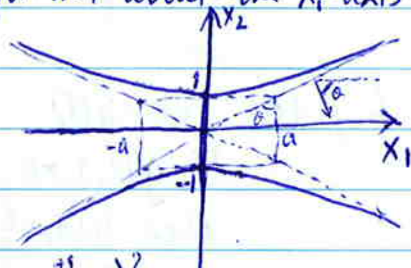
6.2 $n=1$ $N(S) = \{(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})\}$; $n=2$ $N(S) = \{(-\frac{\sqrt{2}}{2}, u, v) \mid u^2+v^2 = \frac{1}{4}\}$

6.3 $n=1$ $N(S) = \{(x_1,x_2) \mid x_1^2 + x_2^2 = 1\}$; $n=2$. $N(S) = \{(x_1\ x_2\ x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$

6.4 $n=1$ $N(S) = \{(x_1,x_2) \mid x_1^2 + x_2^2 = 1, x_1 < 0\}$; $n=2$ $N(S) = \{(x_1,x_2,x_3) \mid \sum_{i=1}^{3} x_i^2 = 1, x_1 < 0\}$

6.5 We only ~~need~~ need to analyze $n=1$, the cases for $n\geq 2$ can be ~~derived by~~ viewed as the surface of revolution obtained by rotating the curve for $n=1$ about the x_1-axis then about (x_1,x_2)-plane ~~axis~~, then about (x_1,x_2,x_3) ...

For $n=1$ $-\frac{x_1^2}{a^2} + x_2^2 = 1$ like the right figure.



The spherical image is 

$\theta = tg^{-1} a$. or formally $\{(x_1,x_2)\in S^1 \mid x_1 \in (\frac{-1}{\sqrt{a^2+1}}, \frac{1}{\sqrt{a^2+1}})\}$

For $n\geq 2$. The spherical image is $\{(x_1,\cdots x_{n+1}) \in S^n \mid x_1 \in (\frac{-1}{\sqrt{a^2+1}}, \frac{+1}{\sqrt{a^2+1}})\}$

When $a\to\infty$, it shrinks to a narrow band.

when $a \to 0$, it extends to the whole S^n