5.7 (a)(b) just write out (c) take $u=(1,0,0),(0,1,0),(0,0,1)$ then get it

5.8 (a) consistent $\Leftrightarrow \det\begin{pmatrix} v\\ w\\ N(p)\end{pmatrix}>0 \Leftrightarrow v\cdot(w\times N(p))>0 \Leftrightarrow N(p)\cdot(v\times w)>0$

(b) Denote $\hat{x} \triangleq x/\|x\|$, consistent $\Leftrightarrow \hat{w}\cdot(N(p)\times\hat{v})>0$

~~[As {v,w} is a basis of S_p, As $\|\hat w\|=1$ so there must exist θ. $\hat w = \cos\theta\,\hat v+\sin\theta\, N(p)\times\hat v$~~

(Proof) As $N(p)\cdot(N(p)\times\hat v)=\det\begin{pmatrix}N(p)\\N(p)\\ \hat v\end{pmatrix}=0$ So $N(p)\times\hat v\in S_p$.

$\hat v\cdot(N(p)\times\hat v)=\det\begin{pmatrix}\hat v\\N(p)\\ \hat v\end{pmatrix}=0$. So $\{N(p)\times\hat v, \hat v\}$ is an orthornormal basis of S_p

As $\|\hat w\|=1$. So there exists θ s.t. $\hat w=\cos\theta\cdot\hat v+\sin\theta\cdot N(p)\times\hat v$

So $\hat w\cdot(N(p)\times\hat v)=\sin\theta$

So $\theta\in(0,\pi) \Leftrightarrow \hat w\cdot(N(p)\times\hat v)>0 \Leftrightarrow \{v,w\}$ is consistent with N

5.9 (a) take $u=(1,0,0,0),(0,1,0,0)\cdots(0001)$ (b) just check

5.10 (a) $\det\begin{pmatrix}v_1\\ \vdots\\ v_n\\ N\end{pmatrix}<0 \Leftrightarrow \det\begin{pmatrix}v_1\\ \vdots \\ v_n\\ -N\end{pmatrix}>0$

(b) let $v=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$, $w=\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}$ $\begin{pmatrix}W\\N\end{pmatrix}=\begin{pmatrix}AV\\N\end{pmatrix}=\begin{pmatrix}A&0\\0&1\end{pmatrix}\begin{pmatrix}V\\N\end{pmatrix}$ where $W=\begin{pmatrix}w_1\\ \vdots\\ w_n\end{pmatrix}$, $V=\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix}$

So $\det\begin{pmatrix}W\\N\end{pmatrix}=\det A\cdot\det\begin{pmatrix}V\\N\end{pmatrix}$, thus consistency of W with N is identical to the consistency of V with N iff $\det A>0$

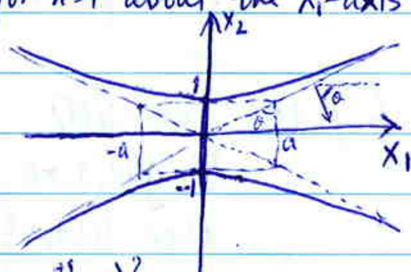
6.1 ~~$N(S)=\{v \mid \|v\| \ldots$~~ $n=1$ $N(S)=\{(0,1),(0,-1)\}$; $n=2$ $N(S)=\{(0,x_2,x_3) \mid x_2^2+x_3^2=1\}$ $\in\mathbb{R}^3$

6.2 $n=1$ $N(S)=\{(-\frac{\sqrt2}{2},\frac{\sqrt2}{2}),(-\frac{\sqrt2}{2},-\frac{\sqrt2}{2})\}$; $n=2$ $N(S)=\{(-\frac{\sqrt2}{2},u,v) \mid u^2+v^2=\frac{1}{2}\}$

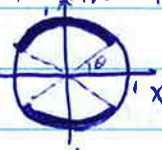
6.3 $n=1$ $N(S)=\{(x_1,x_2) \mid x_1^2+x_2^2=1\}$; $n=2$. $N(S)=\{(x_1,x_2,x_3) \mid x_1^2+x_2^2+x_3^2=1\}$

6.4 $n=1$ $N(S)=\{(x_1,x_2) \mid x_1^2+x_2^2=1, x_1<0\}$; $n=2$ $N(S)=\{(x_1,x_2,x_3) \mid \sum_{i=1}^{3} x_i^2=1, x_1<0\}$

6.5 We only ~~heed~~ need to analyze $n=1$, the cases for $n\geq 2$ can be ~~derived by~~ viewed as the surface of revolution obtained by rotating the curve for $n=1$ about the x_1-axis then about (x_1,x_2)-plane ~~axis~~, then about (x_1,x_2,x_3) $\cdots$



For $n=1$ $-\frac{x_1^2}{a^2}+x_2^2=1$ like the right figure.

The spherical image is 

$\theta=tg^{-1}a$. or formally $\{(x_1,x_2)\in S^1 \mid x_1\in(\frac{-1}{\sqrt{a^2+1}},\frac{1}{\sqrt{a^2+1}})\}$

For $n\geq2$ the spherical image is $\{(x_1,\cdots x_{n+1})\in S^n \mid x_1\in(\frac{-1}{\sqrt{a^2+1}},\frac{+1}{\sqrt{a^2+1}})\}$

When $a\to\infty$, it shrinks to a narrow band

When $a\to0$, it extends to the whole S^n

6.6 Obvious ?

6.7 "if part": Suppose the orientation at p is $N(p)$. Since $\alpha(t) = p+ta \in S$ for all $t \in I$, so $\dot{\alpha}(0) = a \in S_p$. So $a \cdot N(p) = 0$ which is true for any $p \in S$.

"only if" part: Consider the constant vector field $X(q) = (q, a)$. It is a tangent field on S because at $\forall p$, $N(p)\cdot a = 0$. Now $\alpha(t) = p + at$ is an integral curve of X and $\alpha(0) = p \in S$. Then by the corollary to Theorem 1, Chapter 5, $\alpha(t) \in S$ for all $t \in I$ where I is the interval on which $\alpha(t)$ is defined.

6.8 Suppose $N(S) = \{v\}$. Let B be an open ball contained in U (S is a level set on U) and $p \in S \cap B$. Then for $\forall x_0 \in B$ which satisfies $(x_0 - p)\cdot v = 0$, we construct a constant vector field $W(q) = (q, x_0 - p)$, ~~which is the whose~~ Since $N(S)=\{v\}$, the restriction of $W(q)$ on U is a tangent vector field on S. As B is open, there is $\alpha(t) = p + (x_0 - p)t$ ($(-\varepsilon, 1+\varepsilon) \to B$), an integral curve of W, such that $\alpha(0) \in S$. Thus by corollary to Thm 1, ch 5, $\alpha(t) \in S$, and specifically $\alpha(1) = x_0 \in S$. Therefore $\{x \in R^{n+1} : x \cdot v = p \cdot v\} \cap B \subseteq S$.

Next, suppose a new $\alpha : [a,b] \to S$ is a continuous parametrized curve and $\alpha(t) \in B'$ (a new open set) for $t_1 \le t \le t_2$. If $\alpha(t_1)\cdot v < \alpha(t_2)\cdot v$, then for any $b \in (\alpha(t_1)\cdot v, \alpha(t_2)\cdot v)$, due to $\alpha(t)$ being continuous, there exists $t_3 \in (t_1, t_2)$ s.t. $\alpha(t_3)\cdot v = b$. Since $\alpha(t_3) \in S \cap B$, by above argument, we have $\{x \in R^{n+1} : x \cdot v = \alpha(t_3)\cdot v = b\} \cap B \subseteq S$. Therefore $\{x \in B \mid x \cdot v \in (\alpha(t_1)\cdot v, \alpha(t_2)\cdot v)\} \subseteq S$. But ~~that~~ the left hand set is an open set and therefore $N(S) = S^n$ (because S contains an open set), contradicting with $N(S) = \{v\}$. So $\alpha(t_1)\cdot v \ge \alpha(t_2)\cdot v$. Likewise, $\alpha(t_1) v \le \alpha(t_2)\cdot v$.

So $\alpha(t_1)\cdot v = \alpha(t_2)\cdot v$. Since S is connected, any two points on S p, q can be connected by a continuous parametrized curve, and one can find an open set B s.t. $p, q \in S \cap B$, therefore $p \cdot v = q \cdot v$, i.e., all points in S lie on the same plane (or part of a plane).

6.9 (a) Let $g(t) = f(\alpha(t))$. So now we have $g(t_1) = g(t_2) = c$, $g(t) \ne c$ for all $t \in (t_1, t_2)$.
If $g'(t_1) > 0$, $g'(t_2) > 0$. Then there exists $\varepsilon_1 > 0$, s.t. $g'(t) > 0$ $t \in [t_1, \varepsilon_1 + t_1)$
then $g(t_1 + \frac{\varepsilon_1}{2}) - g(t_1) = g'(t_1 + \xi_1)\cdot\frac{\varepsilon_1}{2}$ where $\xi_1 \in [0, \frac{\varepsilon_1}{2}]$. So $g'(t_1 + \xi_1) > 0$,
thus $g(t_1 + \frac{\varepsilon_1}{2}) > g(t_1) = c$. There also exists $\varepsilon_2 > 0$ s.t. $g'(t) > 0$ $t \in (t_2 - \varepsilon_2, t_2]$
then $g(t_2 - \frac{\varepsilon_2}{2}) - g(t_2) = -g'(t_2 - \xi_2)\frac{\varepsilon_2}{2}$, where $\xi_2 \in [0, \frac{\varepsilon_2}{2}]$ so $g'(t_2 - \xi_2) > 0$
thus $g(t_2 - \frac{\varepsilon_2}{2}) < g(t_2) = c$. Then as g is continuous there exists $t \in (t_1 + \frac{\varepsilon_1}{2}, t_2 - \frac{\varepsilon_2}{2}) \subset (t_1, t_2)$
s.t. $g(t) = c$ contraction!

one can choose small enough $\varepsilon_1, \varepsilon_2$, s.t. $t_1 + \frac{\varepsilon_1}{2} < t_2 - \frac{\varepsilon_2}{2}$

~~(a)~~ If $g(t_1)<0$, $g(t_2)<0$, same contradiction occurs. So $g'(t_1)g'(t_2)<0$

(b) If α crosses S for an odd number of times $t_1 \cdots t_n$ then by (a) $g'(t_1)g'(t_n)>0$. With out loss of generality, suppose $g'(t_1)>0$, $g'(t_n)>0$.

Since $g(t_1)=g(t_n)=c$ (and t_1, t_n are two extreme times) So $g(t)<c$ for all $t<t_1$; $g(t)>c$ for all $t>t_n$

However as S is compact ~~and α goes to ∞ in both directions we can~~ and $f: R^{n+1} \to R$ ~~there is a~~ Suppose S is strictly contained in sphere S': $\|x\|^2=r^2$, ~~then pick any $p \in S$ and consider $S_1 f^{-1}(f(p))$~~. Since α goes to ∞ in both directions

There must be t_0, t_{n+1} with $t_0<t_1$, $t_{n+1}>t_n$, such that $\alpha(t_0)$ and $\alpha(t_{n+1}) \in S'$

~~As $f(\alpha(t_0))<c$, $f(\alpha(t_{n+1}))>c$ and f is continuous on S'~~, So $f(\alpha(t_0))<c$, $f(\alpha(t_{n+1}))>c$

As S' is connected (see Ex. 5.1), there is a continuous parametrized curve $\beta(t) \in S'$.

s.t. $\beta(t^1)=\alpha(t_0)$, $\beta(t^2)=\alpha(t_{n+1})$, As f, β are continuous on S',

there must be a $t^3 \in (t^1, t^2)$ s.t. $f(\beta(t^3))=c$.

But $\beta(t^3) \in S'$, So $\beta(t^3) \bar{\in} S$. This is contradiction!

6.10 (a) ~~$\forall \beta(t)$~~. Since $\beta(a) \in O(S)$, there exists a continuous map $\alpha: [0, +\infty) \to R^{n+1}-S$ s.t. $\alpha(0)=\beta(a)$. $\lim_{t\to\infty}\|\alpha(t)\|=\infty$.

For $\forall \beta(t_0)$. Construct curve $\gamma(t) = \begin{cases} \beta(t_0-t) & t \in [0, t_0-a] \\ \alpha(t-t_0+a) & t \in \cancel{(t_0, \infty)}\ (t_0-a, +\infty) \end{cases}$

then $\gamma(t)$ is continuous from $[0,+\infty) \to R^{n+1}-S$. $\gamma(0)=\beta(t_0)$, $\gamma(+\infty)=\alpha(+\infty)=\infty$

t_0 is arbitrary so $\beta(t) \in O(S)$ for all $t \in [a.b]$

(b) ~~Open set. $\forall p \in$~~ (1) Non-empty. As S is a compact n-surface, ~~there~~ we can find a n-sphere with a large enough radius r, which strictly subsumes S

then pick one point on the n-sphere, p, construct continuous map

$\alpha(t)=pt+p$. So $\alpha(0)=p$. $\forall t>0$. $\|\alpha(t)\|=(t+1)r>r$

So $\alpha(t) \in R^{n+1}-S$, $\lim_{t\to\infty}\|\alpha(t)\|=\infty$. So $p \in O(S)$

(2) open set $\forall p \in O(S)$, $p \in R^{n+1}-S$, as $R^{n+1}-S$ is open (due to $S=f^{-1}(c)$ is n-surface and by definition f is smooth). So there exists an ε-ball around p, (p, ε), such that $\forall x \in (p, \varepsilon)$ satisfy $x \in R^{n+1}-S$

we can easily construct a continuous map from p to x, By (a), $x \in O(S)$, $\forall x \in (p, \varepsilon)$

(3) connected: $\forall p, q \in O(S)$, Suppose there is a n-sphere S_1 with radius r ~~st~~ such that p, q, S are all contained in it (S compact, $r>\|p\|$, $r>\|q\|$).

As $p \in O(S)$ there is a continuous map $\alpha_1: [0,+\infty) \to R^{n+1}-S$, $\alpha_1(0)=p$. $\lim_{t\to\infty}\|\alpha_1(t)\|=\infty$.

Suppose ~~$\alpha(t)$~~ $\|\alpha_1(t_1)\|=r$ (i.e. $\alpha_1(t_1) \in S_1$). Likewise, we define $\alpha_2(t)$ and t_2

As S_1 is connected and $S_1 \subset R^{n+1}-S$, there's a curve α_3 on S_1, s.t. $\alpha_3(a)=\alpha_1(t_1)$, ~~$\alpha_3(b)$~~

$\alpha_3(b)=\alpha_2(t_2)$. $\alpha_3(t)\in O(S)$ by (a). So now construct ~~a~~ continuous curve from p to q in $O(S)$:

$$\gamma(t)=\begin{cases}\alpha_1(t) & t\in[0,t_1]\\ \alpha_3(t-t_1+a) & t\in[t_1,t_1+b-a]\\ \alpha_2(t_2-t+t_1+b-a) & t\in(t_1+b-a,\ t_1+b-a+t_2)\end{cases}$$

7.2 $\|\dot\alpha(t)\|=\text{constant}\Rightarrow \frac{d}{dt}\dot\alpha(t)\cdot\dot\alpha(t)=2\ddot\alpha(t)\cdot\dot\alpha(t)=0$, i.e. $\ddot\alpha(t)\perp\dot\alpha(t)$

7.3 Let $s(t)=\int_{t_0}^{t}\|\dot\alpha(t)\|dt$. As $\dot\alpha(t)\neq 0$, so $s(t)$ monotonic increasing so $s(t)$ is invertible. Let $h=s^{-1}$. h is onto by definition $h'=\frac{1}{s'}=\|\dot\alpha(h(t))\|^{-1}>0$

$\dot\beta=\dot\alpha(h(t))\cdot h'(t)=\dot\alpha(h(t))/\|\dot\alpha(h(t))\|$ so β is unit speed

7.5. ~~α of~~ "if part" is by Example 2 in this chapter

"only if": $\alpha(0)=(r\cos b, r\sin b, d)$, which has covered all possible points on cylinder

$\dot\alpha(0)=(-ra\sin b, ra\cos b, c)$. $N_{\alpha(0)}=\pm(\cos b,\sin b,0)$.

So $\dot\alpha(0)$ has covered all possible initial velocity in $S_{\alpha(0)}$

As geodesic is uniquely determined by initial position and initial velocity these are all possible geodesics on cylinder S.

Another proof is by looking at (G) on page 41. $N(x,y,z)=(x,y,0)$

7.6 "if part" is covered by Example 3 in this chapter

"only if": $\alpha(0)=e_1$, $\dot\alpha(0)=ae_2$. Since $e_2\in S_{e_1}$, a allows all norm of velocity

e_1 allows all possible initial position, $\dot\alpha(0)$ allows all possible initial velocity

due to uniqueness of geodesic by initial position and velocity, these are all possible ~~de~~ geodesics on unit n-sphere.

7.7 "if part": $\dot\beta(t)=\dot\alpha(h(t))h'(t)=\dot\alpha(at+b)\cdot a$, $\ddot\beta(t)=a^2\ddot\alpha(at+b)$. As $\alpha(t)$ is geodesic so $\ddot\alpha(t)\in S^{\perp}_{\alpha(t)}$ $\forall t$. So $\ddot\beta(t)\in S^{\perp}_{\alpha(at+b)}=S^{\perp}_{\beta(t)}$ So β is geodesic

"only if": $\ddot\beta(t)=\ddot\alpha(h(t))\cdot(h'(t))^2+\dot\alpha(h(t))h''(t)$ if β is geodesic, $\ddot\beta(t)\in S^{\perp}_{\beta(t)}=S^{\perp}_{\alpha(h(t))}$ $\ni\ddot\alpha(h(t))$

So $\ddot\beta(t)$ and $\ddot\alpha(h(t))$ are parallel

generally, $\dot\alpha$ and $\ddot\alpha$ are not parallel, and $h''(t)$ $h'(t)$ are scalar so we must require $h''(t)=0$ (E.g. $\alpha(t)=\hat e_1\cos t+\hat e_2\sin t$ $\dot\alpha(t)=-\hat e_1\sin t+\hat e_2\cos t$ $\ddot\alpha(t)=-\hat e_1\cos t-\hat e_2\sin t$, $\theta_{\dot\alpha,\ddot\alpha}=0$. So $\dot\alpha$ and $\ddot\alpha$ are never parallel).

So $h(t)=at+b$. We can't see why we need $a>0$. Since at least when $a=0$, β is still geodesic