$\alpha_3(b)=\alpha_2(t_2)$. $\alpha_3(t)\in O(S)$ by (a). So now construct ~~a~~ continuous curve from p to q in $O(S)$:

$$\gamma(t)=\begin{cases}\alpha_1(t) & t\in[0,t_1]\\ \alpha_3(t-t_1+a) & t\in[t_1,t_1+b-a]\\ \alpha_2(t_2-t+t_1+b-a) & t\in(t_1+b-a,\ t_1+b-a+t_2)\end{cases}$$

7.2 $\|\dot\alpha(t)\|=$ constant $\Rightarrow \frac{d}{dt}\dot\alpha(t)\cdot\dot\alpha(t)=2\ddot\alpha(t)\cdot\dot\alpha(t)=0$, i.e. $\ddot\alpha(t)\perp\dot\alpha(t)$

7.3 Let $s(t)=\int_{t_0}^{t}\|\dot\alpha(t)\|dt$. As $\dot\alpha(t)\neq 0$, so $s(t)$ monotonic increasing so $s(t)$ is invertible. Let $h=s^{-1}$. h is onto by definition $h'=\frac{1}{s'}=\|\dot\alpha(h(t))\|^{-1}>0$

$\dot\beta=\dot\alpha(h(t))\cdot h'(t)=\dot\alpha(h(t))/\|\dot\alpha(h(t))\|$ so β is unit speed

7.5. ~~α for~~ "if part" is by Example 2 in this chapter

"only if". $\alpha(0)=(r\cos b, r\sin b, d)$, which has covered all possible points on cylinder

$\dot\alpha(0)=(-ra\sin b, ra\cos b, c)$. $N_{\alpha(0)}=\pm(\cos b,\sin b,0)$.

So $\dot\alpha(0)$ has covered all possible initial velocity in $S_{\alpha(0)}$

As geodesic is uniquely determined by initial position and initial velocity these are all possible geodesics on cylinder S.

Another proof is by looking at (G) on page 41. $N(x,y,z)=(\hat{x,y},0)$

7.6 "if part" is covered by Example 3 in this chapter

"only if" $\alpha(0)=e_1$, $\dot\alpha(0)=ae_2$. Since $e_2\in S_{e_1}$. a allows all norm of velocity e_1 allows all possible initial position, $\dot\alpha(0)$ allows all possible initial velocity due to uniqueness of geodesic by initial position and velocity, these are all possible ~~de~~ geodesics on unit n-sphere.

7.7 "if part": $\dot\beta(t)=\dot\alpha(h(t))h'(t)=\dot\alpha(at+b)\cdot a$ $\ddot\beta(t)=a^2\ddot\alpha(at+b)$ As $\alpha(t)$ is geodesic so

$\ddot\alpha(t)\in S^{\perp}_{\alpha(t)}$ $\forall t$. So $\ddot\beta(t)\in S^{\perp}_{\alpha(at+b)}=S^{\perp}_{\beta(t)}$ So β is geodesic

"only if": $\ddot\beta(t)=\ddot\alpha(h(t))\cdot(h'(t))^2+\dot\alpha(h(t))h''(t)$ if β is geodesic, $\ddot\beta(t)\in S^{\perp}_{\beta(t)}=S^{\perp}_{\alpha(h(t))}\ni\ddot\alpha(h(t))$

So $\ddot\beta(t)$ and $\ddot\alpha(h(t))$ are parallel generally, $\dot\alpha$ and $\ddot\alpha$ are not parallel, and $h''(t)$ $h'(t)$ are scalar

So we must require $h''(t)=0$ (E.g. $\alpha(t)=\hat e_1\cos t+\hat e_2\sin t$ $\dot\alpha(t)=-\hat e_1\sin t+\hat e_2\cos t$

$\ddot\alpha(t)=-\hat e_1\cos t-\hat e_2\sin t$, $\theta_{\dot\alpha,\ddot\alpha}=0$. So $\dot\alpha$ and $\ddot\alpha$ are never parallel).

So $h(t)=at+b$. We can't see why we need $a>0$. Since at least ~~even~~ when $a=0$. β is still geodesic

7.8 (a) $\dot\alpha_\theta(t) = (\dot x_1(t), \dot x_2(t)\cos\theta, \dot x_2(t)\sin\theta)$

$\dot\beta_t(\theta) = (0, -x_2(t)\sin\theta, x_2(t)\cos\theta)$

$\dot\alpha_\theta(t)\cdot\dot\beta_t(\theta) = 0$

(b) $\ddot\alpha_\theta(t) = (\ddot x_1(t), \ddot x_2(t)\cos\theta, \ddot x_2(t)\sin\theta)$

~~Sp~~ $N(p) = \pm ?$ hard to write So must find another way

Notice that $\dot\alpha_\theta(t) \in S_p$, $\dot\beta_t(\theta) \in S_p$ by definition because $\alpha_\theta(t), \beta_t(\theta)$ are both on S.

by (a) $\dot\alpha_\theta(t) \perp \dot\beta_t(\theta)$. So $\dot\alpha_\theta(t), \dot\beta_t(\theta)$ form a basis of S_p $(p = \alpha_\theta(t))$

So one only needs to check that $\ddot\alpha_\theta(t)$ is orthogonal to $\dot\alpha_\theta(t)$ and $\dot\beta_t(\theta)$

~~$\dot\alpha_\theta(t)\cdot\ddot\alpha_\theta(t) = \ddot x_1(t)\dot x_1(t) + \ddot x_2(t)\dot x_2(t)$~~. As $\alpha(t) = (x_1(t), x_2(t))$ has constant speed, by Ex 7.2 $\dot\alpha_\theta(t) \perp \ddot\alpha_\theta(t)$. $\ddot\alpha_\theta(t) \perp \dot\beta_t(\theta)$ is easy to check.

(c) $\ddot\beta_t(\theta) = (0, -x_2(t)\cos\theta, -x_2(t)\sin\theta)$, obviously $\ddot\beta_t(\theta) \perp \dot\beta_t(\theta)$

$\ddot\beta_t(\theta) \perp \dot\alpha_\theta(t) \Leftrightarrow x_2(t)\cdot\dot x_2(t) = 0$ Since $x_2(t) > 0$ $\dot x_2(t) = 0 \Leftrightarrow x_2'(t)/x_1'(t)\Big|_{\alpha(t)} = 0$

7.9 First check $\beta(t) = \alpha(ct)$ is a maximal ~~d~~ geodesic with initial velocity cv: $\beta(0) = \alpha(0)$

$\dot\alpha(ct) = c\cdot\dot\alpha(t)$ ~~= ct~~. So $\dot\alpha(ct)|_{t=0} = c\cdot\dot\alpha(t)|_{t=0} = cv$.

$\ddot\beta(t) = c^2\ddot\alpha(ct)$. As α is geodesic, so $\ddot\alpha(t) \in S_{\alpha(t)}^{\perp}$. So $\ddot\beta(t) \in S_{\alpha(ct)}^{\perp} = S_{\beta(t)}^{\perp}$

So $\beta(t)$ is geodesic. ~~I is easily~~ Since the geodesic with given initial position and velocity ~~given~~ is unique, $\beta(t)$ is ~~what w~~ the maximal geodesic in S with initial velocity cv.

The domain I can be easily taken care of.

7.10 Define $\gamma(t) = \beta(t+t_0)$, then $\gamma(0) = \beta(t_0) = p$, $\dot\gamma(0) = \dot\beta(t_0) = v$. So if $\gamma(t)$ is geodesic, then by uniqueness theorem, $\gamma(t) = \alpha(t)$, i.e. $\beta(t+t_0) = \alpha(t)$, i.e. $\beta(t) = \alpha(t-t_0)$

I is taken care of because α is maximal

7.11 Let $\gamma(t) = \beta(t)$. $\gamma(t_0) = \beta(t_0) = \beta(0)$ ~~, $\gamma(t_0)$~~, $\dot\gamma(t_0) = \dot\beta(t_0) = \dot\beta(0)$. So by Ex 7.10

$\gamma(t) = \beta(t-t_0)$ i.e. $\beta(t) = \beta(t-t_0)$ i.e. $\beta(t+t_0) = \beta(t)$

7.12 (a) complete by Example 3

(b) incomplete $\alpha(t) = (1,0,\cdots,0)\cos t + (0,\cdots,0,1)\sin t$ is geodesic $t \in (-\frac{3\pi}{2}, \frac{\pi}{2})$ but $t \neq \frac{\pi}{2} + 2k\pi$ $k \in \mathbb{Z}$

(c) incomplete $\alpha(t) = (0,1,1) - (0,1,1)t$ $t \neq 1$

(d) complete by Example 2

(e) incomplete $\alpha(t) = (0,1,0)\cos t + (1,0,0)\sin t$ $t \neq \frac{\pi}{2} \pm 2k\pi$ $k \in \mathbb{Z}$