8.1 (b) $(fX)' = (\dot{f}X) - [(\dot{f}X)\cdot N(\alpha(t))]N(\alpha(t))$

$= \dot{f}X + f\dot{X} - [(\dot{f}X + f\dot{X})\cdot N(\alpha(t))]N(\alpha(t))$ (as $X\cdot N(\alpha(t)) = 0$ by X being tangent to S)

$= \dot{f}X + f\dot{X} - f[\dot{X}\cdot N(\alpha(t))]\cdot N(\alpha(t)) = \dot{f}X + fX'$

8.2 ~~$\alpha(t)$ is always in S, i.e. $\vec{a}\cdot\alpha(t) = b$, i.e. $\dot{\alpha}(t) = 0$. $\dot{\alpha}(t) = 0 \in S^{\perp}_{\alpha(t)}$ So~~

~~[illegible]~~ Define Vector field along α, $\vec{V}(t) = \vec{v}$ on S, As $v \in S_p$.

so V is tangent to S (and of course along α), $\dot{V} = 0$, So V is parallel along α. So $P_\alpha(v) = (q, v)$

That means parallel transport in an n-plane is path independent

8.3 When $v_1 = (p, 1, 0, 0)$ By example on page 49, $V_1(t) = \dot{\alpha}(t) = (\cos t, 0, -\sin t)$, $V_1(\pi) = (-1, 0, 0)$

When $v_2 = (p, 0, 1, 0)$ Then the vector field $V_2(t) =$ ~~[illegible]~~ $(0,1,0)$, is parallel to S along α

So $V_2(\pi) = (0, 1, 0)$. As P_α is linear transform, $P_\alpha(v) = (q, -v_1, v_2, 0)$

8.4 Define geodesic $\alpha(t) = p\cos t + \hat{v}\sin t$, $\dot{\alpha}(t) = -p\sin t + \hat{v}\cos t$. $\alpha(0) = p$, $\alpha(\frac{\pi}{2}) = \hat{v}$

$\dot{\alpha}(0)\cdot v = \|v\|$, As $\dot{\alpha}(\frac{\pi}{2})\cdot P_\alpha(\hat{v}) = -p$, $P_\alpha(\hat{v}) = \|v\|$ and $\|P_\alpha(v)\| = \|v\|$, So $P_\alpha(v) = -p\|v\|$ (By corollary on Pg 48)

Likewise define geodesic $\beta(t) = p\sin t + \hat{w}\cos t$ $\quad P_\beta(-p\|v\|) = w$ $\quad \beta(\frac{\pi}{2}) = p$, $\beta(0) = \hat{w}$

So both $\alpha(\frac{\pi}{2})$ and $\beta(\frac{\pi}{2})$ are on $\{x \in S^2_p \mid p\cdot x = 0\}$. we define geodesic

(by example 3 in ch 7, Pg 40) $\nu(t) = \hat{v}\cos t + \sin t\cdot(p\times\hat{v})$, $\nu(0) = \hat{v}$, We now find t_0 s.t. $\nu(t_0) = \hat{w}$

$\hat{v}\cos t_0 + (p\times\hat{v})\sin t_0 = \hat{w} \Rightarrow \hat{v}\cdot\hat{w}\cdot\cos t_0 + p\cdot(\hat{v}\times\hat{w})\sin t_0 = 1$. Let the angle

between $\hat{v}$ and $\hat{w}$ be θ, since $p\perp\hat{v}$, $p\perp\hat{w}$, we have either

$\cos\theta\cos t_0 + \sin\theta\sin t_0 = 1$ or $\cos\theta\cos t_0 - \sin\theta\sin t_0 = 1$. But in whichever

case, there must be a solution t_0 ($t_0 = \theta$ or $-\theta$). Check $V(t) = -p\|v\|$ is parallel along $\nu(t)$:

$\dot{\nu}(t)\cdot p\|v\| = 0 \quad \forall t$, $\|V(t)\| = \text{constant}$,

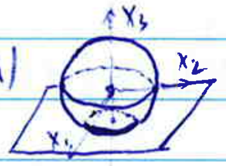
$V(t)\cdot N_{\nu(t)} = \pm p\|v\|\cdot(\hat{v}\cos t + (p\times\hat{v})\sin t) = 0$ So $V(t) \in S_{\nu(t)}$

Therefore $V(t) = -p\|v\|$ is parallel on S^2_p along $\nu(t)$ as $\nu(t)$ is geodesic (by Corollary Pg 48)

So we finally find a piecewise smooth parametrized curve ~~$v \to P_\alpha(v) \to P_\nu(p) \to P_\beta$~~

$v \to P_\alpha(v) = -p\|v\| \to P_\nu(-p\|v\|) = -p\|v\| \to P_\beta(-p\|v\|) = w$.

8.5 (a)



$S_1 = \{(x - (0, 0, \frac{1}{2}))\cdot(0, 0, 1) = 0\}$.

$S_2 = \{x \mid \|x\|^2 = 1\}$. $\alpha(t) = (\frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{2}\sin t, \frac{1}{2})$

$X(t) = (\frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{2}\sin t, 0)$

$X(t)$ is parallel along α as viewed in S_1. But $X(t)$ is not parallel as viewed in S_2 Because $\dot{X}(t) = (-\frac{\sqrt{3}}{2}\sin t, \frac{\sqrt{3}}{2}\cos t, 0) \notin \{\lambda(0, 0, 1) \mid \lambda \in \mathbb{R}\} = S^{\perp}_{\alpha(t)}$

as $S_{1\alpha(t)}^{\perp} = S_{2\alpha(t)}^{\perp} \Leftarrow S_{1\alpha(t)} = S_{2\alpha(t)}$

(b) X is parallel along α in $S_1 \Leftrightarrow X(t) \in S_{1\alpha(t)}^{\perp} \Leftrightarrow X(t) \in S_{2\alpha(t)}^{\perp} \Leftrightarrow X$ is parallel in S_2

(c) α is geodesic in $S_1 \Leftrightarrow \ddot{\alpha}(t) \in S_{1\alpha(t)}^{\perp} \Leftrightarrow \ddot{\alpha}(t) \in S_{2\alpha(t)}^{\perp} \Leftrightarrow \alpha$ is geodesic in S_2

8.6 (a) Let $Y(u) = X(h(u))$

$\dot{X}(t) \in S_{\alpha(t)}^{\perp} \Rightarrow \dot{Y}(u) = \dot{X}(h(u))\, h'(u) \in S_{\alpha(h(u))}^{\perp} = S_{\beta(u)}^{\perp}$.

As $h'(t) \neq 0$. $h(t)$ is monotonic. so there is h^{-1}, $X = Y \circ h^{-1}$. So the same proof as above goes $(h^{-1})' \neq 0$

therefore it is iff.

(b) First by (a), $X \circ h$ is parallel along $\alpha \circ h$. If $\alpha(t_1) = p$, $\alpha(t_2) = q$.

Let $h(u_1) = t_1$, $h(u_2) = t_2$. So $\alpha(h(u_1)) = p$, $\alpha(h(u_2)) = q$.

$X(h(u_1)) = X(t_1)$, $X(h(u_2)) = X(t_2)$. Besides h is monotonic so $h: [u_1, u_2] \to [t_1, t_2]$

Thus, $X \cdot h$ transports p at u_1 to q at u_2

(c) $\forall v$. If $\alpha(t_1) = p$, $\alpha(t_2) = q$ X is parallel to S along α. $X(t_1) = v$. $X(t_2) = u$ (i.e. $P_\alpha(v) = u$)

$\beta(t) \triangleq \alpha(-t)$. As $\dot{X}(t) \in S_{\alpha(t)}^{\perp}$ ~~So $\dot{X}(-t) \in S_{\alpha(t)}^{\perp}$~~ Let $Y(t) = X(-t)$, then

$\dot{Y}(t) = -\dot{X}(-t) \in S_{\alpha(-t)}^{\perp} = S_{\beta(t)}^{\perp}$ ~~So $Y(t)$ is~~ Besides as $X(t) \cdot N_{\alpha(t)} = 0$

$Y(t) \cdot N_{\beta(t)} = X(-t) \cdot N_{\alpha(-t)} = 0$ So $Y(t)$ is parallel along $\beta(t)$

$\beta(-t_2) = q$. $\beta(-t_1) = q$. $Y(-t_2) = X(t_2) = u$. $Y(-t_1) = X(t_1) = v$

So u is transported to v along $\beta(t)$ from q at $-t_2$ to p at $-t_1$, i.e. parallel transport from q to p along $\alpha(-t)$ is the inverse of parallel transport from p to q along α

8.7 (i) γ is α concatenates with β. If P_α, P_β correspond to A and B respectively, then P_γ correspond to $A \cdot B$, which is also nonsingular

(ii) By the third question in Ex 8.6, P_α^{-1} is the parallel transport along $\alpha(-t)$, $\beta(t) \triangleq \alpha(-t)$ $t \in [-b, -a]$, P_β corresponds to A^{-1}

8.8 We use X^* to denote $X'(t)$, the Fermi derivative.

(a) i $(X+Y)^* = (X+Y)' - [(X+Y)'(t) \cdot \dot{\alpha}(t)]\dot{\alpha}(t)$ by $(X+Y)' = X' + Y'$

$= X' - [X'(t) \cdot \dot{\alpha}(t)]\dot{\alpha}(t) + Y' - [Y'(t) \cdot \dot{\alpha}(t)]\dot{\alpha}(t) = X^* + Y^*$

ii $(fX)^* = (fX)' - [(fX)' \cdot \dot{\alpha}(t)]\dot{\alpha}(t)$ (by $(fX)' = f'X + fX'$)

$= (f'X + fX') - [(f'X + fX') \cdot \dot{\alpha}(t)]\dot{\alpha}(t)$ (by $X(t) \cdot \dot{\alpha}(t) = 0$)

$= f'X + f\{X' - [X' \cdot \dot{\alpha}(t)]\dot{\alpha}(t)\} = f'X + fX^*$

iii $(X \cdot Y)' =$ ~~$(X \cdot Y)' - [(XY)' \cdot \dot{\alpha}(t)]\dot{\alpha}(t)$ (by $(XY)' = X'Y + XY'$)~~

~~$= X'Y + XY' - [(X'Y + XY') \cdot \dot{\alpha}(t)]\dot{\alpha}(t)$~~

$= X'Y + Y'X$

by $\dot{\alpha}(t) \cdot Y = \dot{\alpha}(t) \cdot X(t) = 0$

$X^* Y + X \cdot Y^* = [X' - (X'(t) \cdot \dot{\alpha}(t))\dot{\alpha}(t)] Y + X \cdot [Y' - (Y'(t) \cdot \dot{\alpha}(t))\dot{\alpha}(t)] = X'Y + XY'$

(b). By definition, we should have: $X\cdot\dot\alpha=0$, $X\cdot N\circ\alpha=0$ and $X^*=0$

$X^*=0 \Rightarrow \dot X-(\dot X\cdot N\circ\alpha)N\circ\alpha-(\dot X\cdot\dot\alpha)\dot\alpha+(\dot X\cdot N\circ\alpha)(N\circ\alpha\cdot\dot\alpha)\dot\alpha=0$ (*) Note: $\dot\alpha\perp N\circ\alpha$

$X\cdot\dot\alpha=0 \Rightarrow \dot X\dot\alpha+X\cdot\ddot\alpha=0$, $X\cdot N\circ\alpha=0\Rightarrow \dot X\cdot N\circ\alpha+X\cdot N\dot\circ\alpha=0$

Plugging into (*): $\dot X+(X\cdot N\dot\circ\alpha)N\circ\alpha+(X\cdot\ddot\alpha)\dot\alpha=0$. 1st order differential equation together with initial condition $X(t_0)=V$. So there exists a unique solution $X(t)$.

Now check $X\cdot\dot\alpha=0$ and $X\cdot N\circ\alpha=0$

$(X\cdot\dot\alpha)^{\circ}=\dot X\dot\alpha+X\ddot\alpha=X\ddot\alpha-(X\cdot N\dot\circ\alpha)(N\circ\alpha\cdot\dot\alpha)-(X\cdot\ddot\alpha)(\dot\alpha\cdot\dot\alpha)=0$

$(X\cdot N\circ\alpha)^{\circ}=\dot X\cdot N\circ\alpha+X\cdot(N\dot\circ\alpha)=X\cdot(N\dot\circ\alpha)-(X\cdot N\dot\circ\alpha)(N\circ\alpha\cdot N\circ\alpha)-(X\cdot\ddot\alpha)(\dot\alpha\cdot N\circ\alpha)=0$

domain check same as Thm 1 in chapter as $\|X\|$ is constant

(c) (i) F_α is linear map, If V and W are Fermi parallel along α, then so are V+W and cV ($c\in\mathbb{R}$).

(ii) F_α is one to one and onto: the kernel of F_α is zero because $\|F_\alpha(V)\|=0\Leftrightarrow V=0$ by (iii).
So F_α is one-to-one from one n-dim vector space to another. But such maps are onto

(iii) $F_\alpha(V)\cdot F_\alpha(W)=V\cdot W$, $\forall V,W\in\dot\alpha(a)^\perp$ because $(X\cdot Y)^{\circ}=X^*Y+XY^*=0$, i.e. $X\cdot Y$ is constant

9.1 (a) $\nabla f=(4X_1,6X_2)$ $\nabla_v f(p)=(4,0)\cdot(2,1)=8$

(b) $\nabla f=(2X_1,-2X_2)$ $\nabla_v f(p)=(2,-2)\cdot(\cos\theta,\sin\theta)=2(\cos\theta-\sin\theta)$

(c) $\nabla f=(X_2X_3^2, X_1X_3^2, 2X_1X_2X_3)$, $\nabla_v f(p)=(1,1,2)\cdot(a,b,c)=a+b+2c$

(d) $\nabla f=(q,2q)$ $\nabla_v f(p)=$ ~~2q·v~~ $2p\cdot v$

9.2 $\nabla_{e_i}f=\left(\frac{\partial f(p)}{\partial X_1},\cdots\frac{\partial f(p)}{\partial X_{n+1}}\right)\cdot(0,\cdots1,\cdots0)=\frac{\partial f(p)}{\partial X_i}$

9.3 (a) $\nabla X_1=(X_2,X_1)$ $\nabla X_2=(0,2X_2)$ $\nabla_v X=((0,1)\cdot(0,1),(0,0)\cdot(0,1))=(1,0)$

(b) $\nabla X_1=(0,-1)$ $\nabla X_2=(1,0)$ $\nabla_v X=((0,-1)(-\sin\theta,\cos\theta),(1,0)(-\sin\theta,\cos\theta))=(-\cos\theta,-\sin\theta)$

(c) $\nabla X_i=(q,2e_i)$ $\nabla_v X=(2,2,\cdots2)$.

$F(t)=f(\alpha(t))$ $\quad D_v f=F'(t_0)$

9.5 Let $Y(t)=X(\alpha(t))$ be the vector field tangent to S along α. As $D_vX=(X\circ\alpha)'(t_0)$ where $\alpha: I\to S$ is any parametrized curve in S with $\dot\alpha(t_0)=v$. Then quote the properties i-iii in chapter 8 on page 46. Note in (iii) $\nabla_v(X\cdot Y)$ rather than $D_v(X\cdot Y)$ ($\nabla_v X\cdot Y=(X\cdot Y)'$)

9.4 Same as 9.5. $\nabla_vX=(X\circ\alpha)(t_0)$ $\nabla_v f=\dot F$. Then quote the properties i-iii in chapter 7 on Pg 39

9.6. $X(q)\cdot X(q)=1$ By property iii of ch 9 on Pg 54, $\nabla_vX(q)\cdot X(q)=\nabla_v1=0$ i.e. $\nabla_vX\perp X(p)$