13.5 $\left.\begin{array}{l} h(\beta(t)) = c \Rightarrow \nabla h(\beta(t_1)) \cdot \dot\beta(t_1) = 0 \\ \alpha(t_0) = \beta(t_1) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \nabla h(\alpha(t_0)) \cdot \dot\beta(t_1) = 0 \\ \dot\alpha(t_0) = (\operatorname{grad} h)(\alpha(t_0)) \end{array}\right\}$

$\cancel{\Rightarrow (\operatorname{grad} h)(\alpha(t_0)) \cdot \dot\beta(t_1) = 0 \Rightarrow \nabla h}$

$\Rightarrow \dot\alpha(t_0) \cdot \dot\beta(t_1) = (\operatorname{grad} h)(\alpha(t_0)) \cdot \dot\beta(t_1) = \big(\nabla h(\alpha(t_0)) - (\nabla h(\alpha(t_0)) \cdot N(\alpha(t_0)))\, N(\alpha(t_0))\big) \cdot \dot\beta(t_1) = 0$

$= \nabla h(\alpha(t_0)) \cdot \dot\beta(t_1) = 0$ As $N(\alpha(t_0)) \cdot \dot\beta(t_1) = N(\beta(t_1)) \cdot \dot\beta(t_1) = 0$.

14.1 Let $S_1 = f^{-1}(c)$. $S_2 = g^{-1}(d)$, $\alpha(t): I \to S_1$, $\alpha(t_0) = p$. $\dot\alpha(t_0) = v$. As $\varphi(S_1) \subseteq S_2$, $\cancel{g(\alpha(t))}\ g(\varphi(\alpha(t))) = d$

So $\nabla g(\varphi(\alpha(t))) \cdot \dot{\varphi\circ\alpha}(t) = 0$. But $d\varphi(p,v) = \dot{\varphi\circ\alpha}(t_0)$, So $d\varphi(p,v) \perp \nabla g(\varphi(p))$, i.e.

$d\varphi(p,v) \in S_{2\,\varphi(p)}$ So $d\varphi: T(S_1) \to T(S_2)$

14.2 For $\forall p \in U_1$, $v \in R^n$, $d(\psi\circ\varphi)(p,v) = (\psi(\varphi(p)), \nabla f_1(p)\cdot v, \dots, \nabla f_k(p)\cdot v)$, $f_i(p) = \psi_i(\varphi(p))$

$d\varphi(p,v) = (\varphi(p), \nabla\varphi_1(p)\cdot v, \dots \nabla\varphi_m(p)\cdot v)$. Let $u = (\nabla\varphi_1(p)\cdot v, \dots, \nabla\varphi_m(p)\cdot v)$

$d\psi\circ d\varphi(p,v) = (\psi(\varphi(p)), \nabla\psi_1(\varphi(p))\cdot u, \dots \nabla\psi_k(\varphi(p))\cdot u)$

But $\nabla\psi_i(\varphi(p))\cdot u = \sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))(\nabla\varphi_j(p)\cdot v) = \left(\sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))\cdot\nabla\varphi_j(p)\right)\cdot v$ and

$$\nabla f_i(p) = \left(\frac{\partial\psi_i}{\partial x_1} \cdots \frac{\partial\psi_i}{\partial x_m}\right)\begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \cdots & \frac{\partial\varphi_1}{\partial x_n} \\ \frac{\partial\varphi_m}{\partial x_1} & & \frac{\partial\varphi_m}{\partial x_n} \end{pmatrix} = \left(\sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_1}, \dots, \sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_1}\right) = \sum_{j=1}^m \left(\frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_1}, \dots, \frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_n}\right)$$

$$= \sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))\cdot\left(\frac{\partial\varphi_j}{\partial x_1}(p), \dots, \frac{\partial\varphi_j}{\partial x_n}(p)\right) = \sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))\cdot\nabla\varphi_j(p)$$

So $d(\psi\circ\varphi) = d\psi\circ d\varphi$.

14.3. Example 9. $J^T = \begin{pmatrix} -\sin\theta & \cos\theta & \cancel{\sin\phi}\ 0 & \cancel{\cos\phi}\ 0 \\ 0 & 0 & -\sin\phi & \cos\phi \end{pmatrix}$ rank $J = 2$.

Example 10. $J^T = \begin{pmatrix} \cos\frac{\theta}{2}\cos\theta & \cos\frac{\theta}{2}\sin\theta & \sin\frac{\theta}{2} \\ -\sin\theta - \frac{t}{2}\sin\frac{\theta}{2}\cos\theta - t\cos\frac{\theta}{2}\sin\theta, & \cos\theta - \frac{t}{2}\sin\frac{\theta}{2}\sin\theta + t\cos\frac{\theta}{2}\cos\theta, & \frac{t}{2}\cos\frac{\theta}{2} \end{pmatrix} \triangleq \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

$A \triangleq \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = \frac{t}{2}(\cos\theta + \sin^2\theta) + \sin\frac{\theta}{2}\sin\theta$. $B \triangleq \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = \frac{t}{2}\sin\theta(1-\cos\theta) - \sin\frac{\theta}{2}\cos\theta$

If $A = B = 0$ then $(\cos\theta + \sin^2\theta)\frac{-t}{2} = \sin\frac{\theta}{2}\sin\theta$ cross multiply $\times$ we have

$(\cos\theta - 1)\sin\theta\cdot\frac{-t}{2} = \sin\frac{\theta}{2}\cos\theta$

$\frac{t}{2}\sin\frac{\theta}{2}\sin^2\theta(\cos\theta - 1) = \frac{t}{2}\sin\frac{\theta}{2}\cos\theta(\cos\theta + \sin^2\theta)$. i.e. $\frac{t}{2}\sin\frac{\theta}{2} = 0$. So $t = 0$ or $\theta = 2k\pi$ $k \in \mathbb{Z}$

If $t = 0$. $J^T = \begin{pmatrix} \cos\frac{\theta}{2}\cos\theta & \cos\frac{\theta}{2}\sin\theta & \sin\frac{\theta}{2} \\ -\sin\theta & \cos\theta & 0 \end{pmatrix}$, $A^2 + B^2 + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2 = 1$ So rank $J = 2$.

If $\overset{t\neq 0}{\theta} = 2k\pi$, $J^T = \begin{pmatrix} \cos k\pi & 0 & 0 \\ 0 & 1 + t\cos k\pi & \frac{t}{2}\cos k\pi \end{pmatrix}$, $A^2 + B^2 + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}^2 = (1+t)^2 + \frac{1}{4}t^2 > 0$ So rank $J = 2$.

In all, rank $J = 2$ for all t, θ.

14.4 Let $\alpha: I \to R^2$ be a parametrized curve $\alpha(t) = \begin{pmatrix} \alpha_1(t) \\ \alpha_3(t) \end{pmatrix}\begin{matrix} x_1 \\ x_3 \end{matrix}$, then the parametrized surface obtained by rotating about x_3-axis is $(\alpha_1(t)\cos\theta, \alpha_1(t)\sin\theta, \alpha_3(t))$. In Example 4 $\alpha(\phi) = \begin{pmatrix} r\sin\phi \\ r\cos\phi \end{pmatrix}$.

Example 8 $\alpha(\phi) = \begin{pmatrix} a + b\cos\phi \\ b\sin\phi \end{pmatrix}$

14.5 (a) $J^T = \begin{pmatrix} \cos\phi\sin\theta\sin\psi & -\sin\phi\sin\theta\sin\psi & 0 & 0 \\ \sin\phi\cos\theta\sin\psi & \cos\phi\cos\theta\sin\psi & -\sin\theta\sin\psi & 0 \\ \sin\phi\sin\theta\cos\psi & \cos\phi\sin\theta\cos\psi & \cos\theta\cos\psi & -\sin\psi \end{pmatrix} \triangleq (A_1, A_2, A_3, A_4)$

$|A_1 A_2 A_3|^2 + |A_1 A_2 A_4|^2 + |A_1 A_3 A_4|^2 + |A_2 A_3 A_4|^2 = 1 + \sin^2\theta \sin^6\psi > 0$, So rank $J = 3$

(b) $(\sin\phi\sin\theta\sin\psi)^2 + (\cos\phi\sin\theta\sin\psi)^2 + (\cos\theta\sin\psi)^2 + \cos^2\psi = 1$

14.6 $J_\psi = \begin{pmatrix} t_{n+1} J_\varphi & \begin{matrix} -a_1 \\ -a_{n+1} \end{matrix} \\ 0 & -a_{n+2} \end{pmatrix}$ $|J_\psi| = -a_{n+2} t_{n+1} |J_\varphi| \neq 0$ (as $t \neq 0$, $a_{n+2} \neq 0$, $|J_\varphi| \neq 0$ by assumption)

14.7 Let $d\varphi(v) = (\varphi(v), u) = (\varphi(v), \nabla\varphi_1 \cdot v, \cdots, \nabla\varphi_{n+k} \cdot v)$ Let $Y = X \circ \varphi$

$\nabla_v (X\circ\varphi) = (\nabla Y_1 \cdot v, \cdots \nabla Y_{n+k} \cdot v)$ $\nabla_{d\varphi(v)} X = (\nabla X_1 \cdot u, \cdots, \nabla X_{n+k} \cdot u)$

But $\nabla X_i \cdot u = \sum_{j=1}^{n+k} \frac{\partial X_i}{\partial x_j} \nabla\varphi_j \cdot v = \left(\sum_{j=1}^{n+k} \frac{\partial X_i}{\partial x_j} \nabla\varphi_j\right) \cdot v$

$\nabla Y_i = \left(\frac{\partial X_i}{\partial x_1} \cdots \frac{\partial X_i}{\partial x_{n+k}}\right) \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \cdots & \frac{\partial\varphi_1}{\partial x_{n+k}} \\ \frac{\partial\varphi_{n+k}}{\partial x_1} & \cdots & \frac{\partial\varphi_{n+k}}{\partial x_{n+k}} \end{pmatrix} = \sum_{j=1}^{n+k} \frac{\partial X_i}{\partial x_j} \nabla\varphi_j =$ So $\nabla X_i \cdot u = \nabla Y_i \cdot v$

i.e. $\nabla_v (X\circ\varphi) = \nabla_{d\varphi(v)} X$

14.8 (a) $\|N\| = 1$. $\det\begin{pmatrix} E_1 \\ E_2 \\ N(p) \end{pmatrix} = \|E_1 \times E_2\| > 0$ as E_1, E_2 are linearly independent for parametrized 2-surface.

$N \perp E_1$, $N \perp E_2$, E_1, E_2 form a basis for $d\varphi_p$. So $N \perp$ Image $d\varphi_p$. So N is orientation vector field

As for uniqueness. $N \perp E_1$, $N \perp E_2 \Rightarrow N = \lambda \cdot E_1 \times E_2$, then $\|N\| = 1 \Rightarrow \lambda = \pm E_1 \times E_2$.

then $\det\begin{vmatrix} E_1 \\ E_2 \\ N \end{vmatrix} > 0 \Rightarrow N = E_1 \times E_2 / \|E_1 \times E_2\|$.

(b) E_1, E_2 are smooth wrt p as they are just the i-th column of ~~Jabo~~ Jacobian, So N is smooth

14.9 (a) Look at matrix $A = \begin{pmatrix} E_1 \\ \vdots \\ E_n \\ X \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n+1} \\ E_{n1} & E_{n2} & \cdots & E_{nn+1} \\ X_1 & X_2 & \cdots & X_{n+1} \end{pmatrix}$ Let $A_i = \begin{pmatrix} E_{11} & \cdots & E_{1,i-1} & E_{1,i+1} & \cdots & E_{1,n+1} \\ \vdots \\ E_{n1} & \cdots & E_{n,i-1} & E_{n,i+1} & \cdots & E_{n,n+1} \end{pmatrix}$ $i = 1, 2, \cdots, n+1$ ~~So det A =~~ ~~Σ det A~~

So $\det A = \sum_{i=1}^{n+1} (-1)^{n+1+i} \det A_i X_i = \sum_{i=1}^{n} (\det A_i)^2$ as $X_i = (-1)^{n+1+i} \det A_i$.

So $\det A \geq 0$ and $\det A = 0$ iff $\det A_i = 0$ for all $i = 1 \cdots n+1$. But that contradicts the fact that φ is a parametrized n-surface, i.e. Jacobian is non-singular. So $\det A > 0$

If $X(p) = 0$, then $|A| = 0$ which is impossible. Hence $X(p) \neq 0$ for all $p \in U$.

(b) For $i = 1 \cdots n$, $E_i \cdot X = \det\begin{pmatrix} E_1 \\ \vdots \\ E_i \end{pmatrix} = 0$ So $X \perp E_i$ So $X \perp$ Image $d\varphi_p$. So X is normal vector field along φ

(c) Combining (b), $\det A > 0$, and $\|N\| = 1$. We have N is orientation vector field along φ.

(d) X_i is ~~continuous~~ smooth and $X(p) \neq 0$. So N is smooth.

14.10 $E_i(p) = (\varphi(p), 0, \cdots, 1, \cdots 0, \frac{\partial g}{\partial u_i}(p))$. So $E_i(p) \cdot N(p) = 0$ Let $A = \begin{pmatrix} E_1 \\ E_n \\ N \end{pmatrix}$ ~~then~~ before normalization let $a_i = \frac{\partial g}{\partial u_i}(p)$, then

$\det A = \begin{bmatrix} 1 & & a_1 \\ & \ddots & a_n \\ -a_1 & \cdots & -a_n & 1 \end{bmatrix} = 1 + \sum_{i=1}^{n} a_i^2 > 0$. $\|N\| = 1$ So N is orientation vector field along φ

14.11 Proof is essentially similar to proving Thm2 in Chapter 9. Let $v, w \in R_p^n$ and orientation N

We need to prove $L_p(d\varphi_p(v))\cdot d\varphi_p(w) = L_p(d\varphi_p(w))\cdot d\varphi_p(v)$ i.e. $+\nabla_v N\cdot d\varphi_p(w) = \nabla_w N d\varphi_p(v)$

Let X be the one defined in Ex 14.9. then

$$\nabla_v N\cdot d\varphi_p(w) = \nabla_v\left(\frac{X}{\|X\|}\right)\cdot d\varphi_p(w) = \left[(\nabla_v X)\frac{1}{\|X\|} \cancel{d\varphi_p(w)} + \left(\nabla_v \frac{1}{\|X\|}\right)X\right] d\varphi_p(w) = \frac{1}{\|X\|}\nabla_v X\cdot d\varphi_p(w) = v^T J_X^T(p) J_{\varphi}^{(p)} w/\|X(p)\|$$

Similarly $\nabla_w N\cdot d\varphi(v) = w^T J_N^T(p) J_\varphi(p) v/\|X(p)\|$ So we only need to prove that $J = J_X^T J_\varphi$ is symmetric.

But $\cancel{J_{ij}}$ $J_{ij} = \sum_{k=1}^{n+1} \frac{\partial X_k}{\partial x_i}\frac{\partial \varphi_k}{\partial x_j}$ $\quad J_{ji} = \sum_{k=1}^{n+1} \frac{\partial X_k}{\partial x_j}\frac{\partial\varphi_k}{\partial x_i}$. Let $J_\varphi = (J_{\varphi_1}, J_{\varphi_2},\dots,J_{\varphi_n})$. by def Ex 14.9.

$X\perp J_{\varphi i}$ $(i=1\cdots n)$ So $X\cdot J_{\varphi i}=0$ Taking derivative $J_X^T J_{\varphi i} + H_{\varphi i}^T X = 0$ where

$$H_{\varphi i} = \begin{pmatrix} \frac{\partial^2\varphi_1}{\partial x_i\partial x_1} & \cdots & \frac{\partial^2\varphi_1}{\partial x_i \partial x_n}\\ \vdots & & \vdots \\ \frac{\partial^2\varphi_{n+1}}{\partial x_i\partial x_1} & \cdots & \frac{\partial^2\varphi_{n+1}}{\partial x_i\partial x_n}\end{pmatrix}$$

So we have for $j=1\cdots n$. $\sum_{k=1}^{n+1} \frac{\partial X_k}{\partial x_j}\frac{\partial \varphi_k}{\partial x_i} + \sum_{k=1}^{n+1} \frac{\partial^2\varphi_k}{\partial x_i\partial x_j}X_k = 0$

Similarly: $\sum_{k=1}^{n+1} \frac{\partial X_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j} + \sum_{k=1}^{n+1}\frac{\partial^2\varphi_k}{\partial x_j\partial x_i}X_k=0$.

As $\frac{\partial^2\varphi_k}{\partial x_i\partial x_j} = \frac{\partial^2\varphi_k}{\partial x_i\partial x_j}$ So $\sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_j}\frac{\partial\varphi_k}{\partial x_i} = \sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$, i.e. $J_{ij}=J_{ji}$.

14.12 $d\varphi(p,v) = (\varphi(p), \nabla_v\varphi_1, \cdots \nabla_v\varphi_{n+k})$, $J_{d\varphi} = \begin{pmatrix} J_\varphi(p) & 0 \\ \text{sth.} & \begin{matrix}\nabla\varphi_1 \\ \vdots \\ \nabla\varphi_{n+k}\end{matrix}\end{pmatrix} = \begin{pmatrix}J_\varphi(p) & 0\\ \text{sth} & J_\varphi(p)\end{pmatrix}$ As $J_\varphi(p)$ is fullranked, $J_{d\varphi(p,v)}$ must be fullranked as well

14.13 $\nabla_{e_i}E_j = \nabla_{e_i}\left(\varphi(p), \frac{\partial\varphi_1}{\partial x_j}, \cdots \frac{\partial\varphi_{n+k}}{\partial x_j}\right) = \left(\frac{\partial^2\varphi_1}{\partial x_i\partial x_j}, \frac{\partial^2\varphi_2}{\partial x_i\partial x_j},\dots,\frac{\partial^2\varphi_{n+k}}{\partial x_i\partial x_j}\right) = \nabla_{e_j}E_i$

14.14 (a) Let $\begin{pmatrix}L_p(E_1(p))\\ \vdots \\ L_p(E_n(p))\\ N(p)\end{pmatrix} = A^T\cancel{\begin{pmatrix}E_1(p)\\ \vdots\\ E_n(p)\end{pmatrix}}$, $\begin{pmatrix}A^T & \\ & 1\end{pmatrix}\begin{pmatrix}E_1(p)\\ \vdots\\ E_n(p)\\ N(p)\end{pmatrix}$ By definition of $N(p)$, $\begin{pmatrix}E_1(p)\\ E_n(p)\\ N(p)\end{pmatrix}\neq 0$, so

$\det|A| = \det\begin{pmatrix}L_p(E_1(p))\\ L_p(E_n(p))\\ N(p)\end{pmatrix}/\det\begin{pmatrix}E_1(p)\\ E_n(p)\\ N(p)\end{pmatrix}$, as $L_p(E_i(p)) = -\nabla_{e_i}N = -\left(\varphi(p), \frac{\partial N_1}{\partial x_i}(p),\dots,\frac{\partial N_{n+1}}{\partial x_i}(p)\right)$

$= (-1)^n \det\begin{pmatrix}J_N^T\\N\end{pmatrix}/\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix}$ (1)

On the other hand $L_p[E_i(p)]\cdot E_j(p) = -\nabla_{e_i}N\cdot E_j(p) = -\left(\frac{\partial N_1}{\partial x_i}\cdots\frac{\partial N_{n+1}}{\partial x_i}\right)\cdot E_j(p) = -\sum_{k=1}^{n+1} \frac{\partial N_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$ (*)

$E_i(p)\cdot E_j(p) = \sum_{k=1}^{n+1} \frac{\partial\varphi_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$

So $\det(L_p[E_i(p)]\cdot E_j(p))/\det(E_i(p)\cdot E_j(p)) = (-1)^n\det(J_N^TJ_\varphi)/\det(J_\varphi^TJ_\varphi)$ (2)

Notice $\begin{pmatrix}J_N^T\\N\end{pmatrix}(J_\varphi\ N^T) = \begin{pmatrix}J_N^TJ_\varphi & J_N^TN^T\\ NJ_\varphi & NN^T\end{pmatrix}$ By definition of N, $NN^T=1$. $NJ_\varphi=0$ we have

$\det\begin{pmatrix}J_N^T\\N\end{pmatrix}\cdot\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix} = \det(J_N^TJ_\varphi)$ (3)

Likewise $\begin{pmatrix}J_\varphi^T\\N\end{pmatrix}(J_\varphi\ N^T) = \begin{pmatrix}J_\varphi^TJ_\varphi & J_\varphi^TN^T\\ NJ_\varphi & NN^T\end{pmatrix}$ hence $\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix}\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix} = \det(J_\varphi^TJ_\varphi)$ (4)

$\frac{(3)}{(4)}$ we have $\det\begin{pmatrix}J_N^T\\N\end{pmatrix}/\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix} = \det(J_N^TJ_\varphi)/\det(J_\varphi^TJ_\varphi)$ then by (1), (2) we prove

$K(p) = \det A = \det[L_p[E_i(p)]\cdot E_j(p)]/\det(E_i(p)\cdot E_j(p))$

(b) $\cancel{\nabla_{e_i}E_j = \nabla_{e_i}\left(\frac{\partial\varphi_1}{\partial x_j} \cdots \frac{\partial\varphi_{n+1}}{\partial x_j}\right) = \left(\frac{\partial^2\varphi_1}{\partial x_j\partial x_i}\cdots\frac{\partial^2\varphi_{n+1}}{\partial x_j\partial x_i}\right)}$ As $E_j\cdot N=0$ we have

$0 = \nabla_{e_i}(E_j N) = \nabla_{e_i}E_j N + E_j(p)\nabla_{e_i}N$, So $\nabla_{e_i}E_j\cdot N(p) = (-\nabla_{e_i}N)\cdot E_j(p) = L_p(E_i(p))\cdot E_j(p)$

So $\det[L_p[E_i(p)]\cdot E_j(p)]/\det(E_i(p)\cdot E_j(p)) = \det[\nabla_{e_i}E_j\, N(p)]/\det(E_i(p)\cdot E_j(p))$.

So if n = even number, then whether using N or $-N$ doesn't matter

For Ex 14.15 - 14.18, there's no need to check N or $-N$.

14.15 $J\varphi = \begin{pmatrix} -a\sin\theta\sin\phi & a\cos\theta\cos\phi \\ a\cos\theta\sin\phi & a\sin\theta\cos\phi \\ 0 & -a\sin\phi \end{pmatrix} = (E_1, E_2)$, $N = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

$L_p(E_1(p)) = \nabla_{(1,0)} N = -\frac{\partial N}{\partial\theta} = (\sin\theta\sin\phi, -\cos\theta\sin\phi, 0) = \frac{-1}{a}E_1(p)$

$L_p(E_2(p)) = -\nabla_{(0,1)}N = -\frac{\partial N}{\partial\phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi) = \frac{1}{a}E_2$. So $k(p) = \frac{-1}{a^2}$

14.16 $J\varphi = \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \\ 1 & 0 \end{pmatrix}$ $N = (\cos\theta, \sin\theta, 0)$ $L_p(E_1(p)) = (0,0,0) = 0\cdot E_1(p)$ So $k(p) = 0$

14.17 ~~$L_p(E_i$~~ $J\varphi = \begin{array}{c} \begin{array}{cc} E_1 & E_2 \end{array} \\ \begin{pmatrix} \cos\theta & -t\sin\theta \\ \sin\theta & t\cos\theta \\ 0 & 1 \end{pmatrix} \end{array}$ $N = (\sin\theta, -\cos\theta, t)/\sqrt{t^2+1}$

$L_p(E_1(p)) = (-\sin\theta\, t(t^2+1)^{-\frac{3}{2}}, \cos\theta\, t(t^2+1)^{-\frac{3}{2}}, (t^2+1)^{-\frac{3}{2}})$

$L_p(E_2(p)) = (\cos\theta(t^2+1)^{-\frac{1}{2}}, \sin\theta(t^2+1)^{-\frac{1}{2}}, 0)$ $\det[E_i(p), E_j(p)] = t^2+1$.

$\det[L_p(E_i(p))\cdot E_j(p)] = \begin{vmatrix} 0 & (t^2+1)^{-1/2} \\ (t^2+1)^{-1/2} & 0 \end{vmatrix} = -(t^2+1)^{-1}$ So $k(p) = -(t^2+1)^{-2}$

14.18 $J\varphi = \begin{pmatrix} \cosh t & 0 \\ \sinh t\cos\theta & -\cosh t\sin\theta \\ \sinh t\sin\theta & \cosh t\cos\theta \end{pmatrix}$ $N = (\sinh t, -\cosh t\cos\theta, -\cosh t\sin\theta)/\sqrt{\cosh 2t}$

Using the fact that $\nabla_{e_i}E_j = \left(\frac{\partial^2\varphi_1}{\partial x_i\partial x_j}, \cdots, \frac{\partial^2\varphi_{n+1}}{\partial x_i\partial x_j}\right)$,

we have $\nabla_{e_1}E_1 = (\sinh t, \cosh t\cos\theta, \cosh t\sin\theta)$ $\nabla_{e_1}E_2 = \nabla_{e_2}E_1 = (0, -\sinh t\sin\theta, \sinh t\cos\theta)$

$\nabla_{e_2}E_2 = (0, -\cosh t\cos\theta, -\cosh t\sin\theta)$

$\det[E_i(p)\cdot E_j(p)] = \cosh 2t\cdot\cosh^2 t$. $\det[\nabla_{e_i}E_j\cdot N(p)] = -\cosh^2 t/\cosh 2t$.

So $k(p) = -(\cosh 2t)^{-2}$

14.19 $J\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x & 2y & 2z \end{pmatrix}$ $N = (2x, 2y, 2z, -1)/\sqrt{1+4(x^2+y^2+z^2)}$. ~~$L_p(E_i(p)) = \frac{\partial N}{\partial}$~~ $H\varphi_1 = H\varphi_2 = H\varphi_3 = 0$

$H\varphi_4 = \mathrm{diag}(2,2,2)$ So $\nabla_{e_i}E_i = (0,0,0,2)$ for $i=1,2,3$. $\nabla_{e_i}E_j = (0,0,0,0)$ for $i \neq j$.

$\det[E_i(p)\cdot E_j(p)] = \begin{vmatrix} 1+4x^2 & 4xy & 4xz \\ 4xy & 1+4y^2 & 4yz \\ 4xz & 4yz & 1+4z^2 \end{vmatrix} = 1+4(x^2+y^2+z^2)$.

$\det[\nabla_{e_i}E_j\cdot N(p)] = -8(1+4(x^2+y^2+z^2))^{-3/2}$, So $k(p) = -8(1+4(x^2+y^2+z^2))^{-5/2}$

14.20(a) $J\varphi = \begin{pmatrix} x' & 0 \\ y'\cos\theta & -y\sin\theta \\ y'\sin\theta & y\cos\theta \end{pmatrix}$ $N = (y', -x'\cos\theta, -x'\sin\theta)/(y'^2+x'^2)^{1/2}$

$H\varphi_1 = \begin{pmatrix} x'' & 0 \\ 0 & 0 \end{pmatrix}$, $H\varphi_2 = \begin{pmatrix} y''\cos\theta & -y'\sin\theta \\ -y'\sin\theta & -y\cos\theta \end{pmatrix}$, $H\varphi_3 = \begin{pmatrix} y''\sin\theta & y'\cos\theta \\ y'\cos\theta & -y\sin\theta \end{pmatrix}$

$\nabla_{e_1}E_1 = (x'', y''\cos\theta, y''\sin\theta)$ $\nabla_{e_1}E_2 = \nabla_{e_2}E_1 = (0, -y'\sin\theta, y'\cos\theta)$. $\nabla_{e_2}E_2 = (0, -y\cos\theta, -y\sin\theta)$

So $\det[E_i(p)\cdot E_j(p)] = \begin{vmatrix} x'^2+y'^2 & 0 \\ 0 & y^2 \end{vmatrix} = y^2(x'^2+y'^2)$

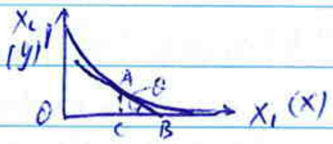
$\det[\nabla_{e_i}E_j\cdot N(p)] = \begin{vmatrix} x''y'-x'y'' & 0 \\ 0 & x'y \end{vmatrix} / (x'^2+y'^2) = (x''y'-x'y'')x'y/(x'^2+y'^2)$

So $k(p) = x'(x''y'-x'y'')/y(x'^2+y'^2)^2$

(b) If $\|\alpha(t)\| = 1$, then $x'^2+y'^2 = 1$ $\ddot{\alpha}\cdot\dot{\alpha} = 0$, i.e. $x''x'+y''y' = 0$

So $x'x''y' = -y''y'^2 = -y''(1-x'^2)$. So $k(p) = \frac{1}{y}(-y''+y''x'^2-x'^2y'') = \frac{-1}{y}y''$

14.21 (a) $x'^2+y'^2 = 1-e^{-2t}+e^{-2t} = 1 = \|\dot\alpha(t)\|^2 = \|\dot\alpha(t)\|$



(b) $-\tan\theta = y'/x' = -e^{-t}/\sqrt{1-e^{-2t}}$. So $\sin\theta = e^{-t}$

$|AB| = y/\sin\theta = e^{-t}/e^{-t} = 1$.

(c) $k = -y''/y = -e^{-t}/e^{-t} = -1$ by Ex 14.20(b) and α being unit speed.

15.1 For $(x_1, x_2, \cdots x_n, 0) \in R^n$. (Equatorial hyperplane). Solve

$\|t(x_1,\cdots,x_n,0) + (1-t)(0,\cdots,0,-1)\| = 1$, i.e. $\|(tx_1,\cdots,tx_n,t-1)\| = 1$

So $t^2(x_1^2+\cdots+x_n^2) + (t-1)^2 = 1$. If $t \neq 0$, then $t = 2\left(\sum_{i=1}^n x_i^2 + 1\right)^{-1}$

So $\varphi(x_1,\cdots x_n,0) = \left(2x_1,\cdots,2x_n, 1-\sum_{i=1}^n x_i^2\right) / \left(\sum_{i=1}^n x_i^2+1\right)$

15.2 For $(x_1\cdots x_n,-1)$. Solve $\|(1-t)(0,0,\ldots,0,1)+t(x_1\cdots x_n,-1)\| = 1$, so $t = 4\left(4+\sum_{i=1}^n x_i^2\right)^{-1}$

So $\varphi(x_1\cdots x_n,-1) = \left(4x_1,\cdots,4x_n, \sum_{i=1}^n x_i^2 - 4\right) / \left(\sum_{i=1}^n x_i^2 + 4\right)$

15.3 (a) If $v(t) \in f^{-1}(c)$. let $(\alpha(t), s(t)) = \psi|_V^{-1}(v(t))$. so

$f(v(t)) = f(\psi(\alpha(t),s(t))) = s(t) = c$. So $v(t) = \psi(\alpha(t),c) = \varphi\circ\alpha + c\cdot N\circ\alpha$.

If $\beta_q(s) = v(t)$, i.e. $\varphi(q) + sN(q) = \varphi(\alpha(t)) + cN(\alpha(t))$. then since there is a smooth inverse of $\psi|_V$, so $q = \alpha(t)$. $s = c$. then

$v'(t)\cdot\beta_q'(s) = N(q)\cdot((\dot\varphi\circ\alpha)(t) + c\cdot(\dot N\circ\alpha)(t)) = N(\alpha(t))((\dot\varphi\circ\alpha)(t) + c\cdot(\dot N\circ\alpha)(t))$

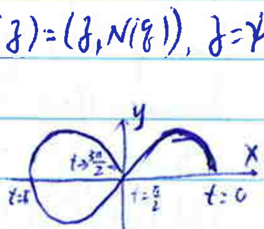
As $\|N(\alpha(t))\| \equiv 1$ So $N(\alpha(t))\cdot\dot N\circ\alpha(t) = 0$. By definition, $(N\circ\alpha)(t)\cdot(\dot\varphi\circ\alpha)(t) = 0$

So $v'(t)\cdot\beta_q'(s) = 0$. i.e. $f^{-1}(c)$ are everywhere orthogonal to the lines $\beta_q(s)$.

(b) By (a) the vector part of $\nabla f(\psi(q,s)) = \lambda\cdot\beta_q'(s) = \lambda N(q)$.

But $\frac{\partial f}{\partial s} = 1$, i.e. $\nabla f\cdot\frac{\partial\psi}{\partial s} = \nabla f\cdot N(q) = 1$ So $\lambda = 1$ $\nabla f(g) = (g, N(q))$, $g = \psi(q,s)$

15.4 $(x(t), y(t)) = (2\cos t, \sin 2t)$ $t \in (0, \frac{3\pi}{2})$



$(x'(t), y'(t)) = (-2\sin t, 2\cos 2t) \neq (0,0)$ obviously one to one

But when $t \to \frac{3\pi}{2}$, the curve approaches its own point $(0,0)$ crossed at $t = \frac{\pi}{2}$, so not n-surface.

15.5 $\forall(\overset{R^{n+1}}{p}, \overset{R^{n+1}}{v}) \in T(S)$. $f(p) = c$, $v\cdot N(p) = 0$. $J = \begin{pmatrix} \nabla f^T & 0 \\ sth. & N(p) \end{pmatrix} = \nabla f\cdot N(p) \neq 0$.

So $T(S)$ is $2n$-surface in R^{2n+2}

15.6 $\forall(p,v) \in T(S)$. $f(p) = c$, $v\cdot N(p) = 0$ $v\cdot v = 0$ $J = \begin{pmatrix} \nabla f^T & 0 \\ \beta & N(p) \\ 0 & 2v \end{pmatrix}$. If $\alpha_1\begin{pmatrix}\nabla f\\ 0\end{pmatrix} + \alpha_2\begin{pmatrix}\beta\\ N(p)\end{pmatrix} + \alpha_3\begin{pmatrix}0\\ 2v\end{pmatrix} = 0$

then $\left.\begin{array}{l}\alpha_1\nabla f + \alpha_2\beta = 0\\ \alpha_2 N(p) + \alpha_3\cdot 2v = 0\end{array}\right\} \Rightarrow \alpha_2 = -2v\cdot N(p) = 0 \Rightarrow \alpha_3 = 0 \Rightarrow \alpha_1 = 0$ So independent, thus $T_1(S)$ is $(2n-1)$-surface in R^{2n+2}

15.7 (a) To be in $O(2)$, the matrix must satisfy: $x_1^2+x_2^2-1=0$, $x_1x_3+x_2x_4=0$, $x_3^2+x_4^2-1=0$

$$J=\begin{pmatrix} 2x_1 & 2x_2 & 0 & 0\\ x_3 & x_4 & x_1 & x_2\\ 0 & 0 & 2x_3 & 2x_4\end{pmatrix} = (J_1\ J_2\ J_3\ J_4)$$ ~~If $(\alpha_1,\alpha_2,\alpha_3)J=0$~~

$Q=(\det(J_1J_2J_3))^2+(\det(J_1J_3J_4))^2+(\det(J_2J_3J_4))^2+(\det(J_1J_2J_4))^2$

$=16(x_1x_4-x_2x_3)^2\sum_{i=1}^{4}x_i^2$ if $=0$ So ① $\sum x_i^2=0 \Rightarrow x_i=0 \Rightarrow x_1^2+x_2^2-1\neq 0$ contradiction!

② $x_1x_4=x_2x_3$, So $x_1x_4x_3=x_2x_3^2$, i.e. $-x_2x_4^2=x_2x_3^2$ So $x_2=0\Rightarrow x_1=\pm1\Rightarrow x_4=0$

$\Rightarrow x_3=\pm1 \Rightarrow x_1x_3=\pm1$ but $x_2x_4=0 \Rightarrow x_1x_3+x_2x_4\neq 0$ contradiction

So $Q\neq 0$, rank$(J)=3$, $O(2)$ is 1-surface in R^4.

(b) Now $J=\begin{pmatrix}2&0&0&0\\0&1&1&0\\0&0&0&2\end{pmatrix}$ $O(2)_p=\{(a,b,c,d)\mid J\cdot\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}2&0&0&0\\0&1&1&0\\0&0&0&2\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=0\}=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\mid a=d=0,\ b+c=0\}$

Solution 2 Let $\alpha(t)=\begin{pmatrix}\alpha_1(t)\\ \alpha_2(t)\end{pmatrix}$, then $\alpha(t)\in O(2)\Leftrightarrow \|\alpha_i(t)\|=1$, $\alpha_i(t)\cdot\alpha_j(t)=0$.

So $\alpha_i'\cdot\alpha_i=0$ So $(a\ b)\begin{pmatrix}1\\0\end{pmatrix}=0\Leftrightarrow a=0$. $(c\ d)\begin{pmatrix}0\\1\end{pmatrix}=0\Leftrightarrow d=0$. $\alpha(t_0)=\begin{pmatrix}\alpha_1(t_0)\\ \alpha_2(t_0)\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

$\alpha_i'\cdot\alpha_j+\alpha_i\cdot\alpha_j'=0 \Leftrightarrow (a\ b)\begin{pmatrix}0\\1\end{pmatrix}+(c\ d)\begin{pmatrix}1\\0\end{pmatrix}=0\Leftrightarrow b+c=0$ Let $\alpha'(t_0)=\begin{pmatrix}a&b\\c&d\end{pmatrix}$

So $O(2)_p=\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\mid a=d=0,\ b+c=0\}$.

15.8 (a) Prove that J has rank $\frac12 n(n+1)$ by induction on n. For $n=2$ 15.7 has proven it.
Let the $n\times n$ matrix be written as $\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_n\end{pmatrix}$, the constraints are $\|\alpha_i\|^2=1$, $\alpha_i\alpha_j=0\ (i\neq j)$. So Jacobian is

rank J_n = rank $\begin{pmatrix}2\alpha_1 & & & \\ & 2\alpha_2 & & \\ & & \ddots & 2\alpha_n\\ \alpha_2 & \alpha_1 & & \\ \alpha_n & \cdots & & \alpha_1\\ & & \alpha_n & \alpha_{n-1}\end{pmatrix}$ = rank $\begin{pmatrix}J_{n-1} & 0\\ 0 & 2\alpha_n\\ \alpha_n & \alpha_1\\ \vdots & \vdots\\ \alpha_n & \alpha_{n-1}\end{pmatrix}$ $\frac{n(n+1)}{2}$ rows, n^2 columns

Note the lowest n rows are linearly independent. As If $\exists\ \beta_1\cdots\beta_n\in R$ s.t.

$\beta_n\begin{pmatrix}0\\ \vdots\\ 0\\ \alpha_n^T\end{pmatrix}+\beta_1\begin{pmatrix}\alpha_n^T\\ 0\\ \vdots\\ 0\\ \alpha_1^T\end{pmatrix}+\beta_2\begin{pmatrix}0\\ \alpha_n^T\\ \vdots\\ 0\\ \alpha_2^T\end{pmatrix}+\dots+\beta_{n-1}\begin{pmatrix}0\\ \vdots\\ 0\\ \alpha_n^T\\ \alpha_{n-1}^T\end{pmatrix}=0$ So $\beta_i\alpha_i^T=0$ $i=1..n-1$ ①

$\sum_{i=1}^{n}\beta_i\alpha_i^T=0$ ②

As none of the α_i is straight 0, $\beta_i=0$ for $i=1..n-1$ by ①. Then by ② $\beta_n\alpha_n^T=0$ So $\beta_n=0$.
Finally the rows in $(J_{n-1}\ 0)$ (the first $\frac{n(n-1)}{2}$ rows) are independent of the last n rows, because these $\frac{n(n-1)}{2}$ rows all have last n elements straight 0 and none of α_i is straight 0. ($\alpha_1,\cdots,\alpha_n$ are linearly independent) So rank $J_n=n+$ rank $J_{n-1}=n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$. So $O(n)$ is $\frac{n(n-1)}{2}$ surface in R^{n^2}

(b) Let $\beta=\begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix}\in O(n)_p$ then $J\beta=0$, i.e. $\begin{cases}\alpha_i\cdot\beta_i=0 \Rightarrow \beta_{ii}=0\\ \alpha_i\beta_j+\alpha_j\beta_i=0\Rightarrow \beta_{ji}+\beta_{ij}=0\end{cases}\Rightarrow O(n)_p=\{P\in R^{n\times n}\mid \forall i,j,\ P_{ij}+P_{ji}=0\}$

If we use the hint in Ex 15.7(b). $\alpha(t)\in O(n)$, $\|\alpha_i(t)\|=1$, $\alpha_i(t)\cdot\alpha_j(t)=0$) This is because if $\sum_{i}\beta_i\alpha_i=0$, $\beta_i\in R$, then $0=\alpha_i\cdot\sum\beta_j\alpha_j=\beta_i\alpha_i\alpha_i=\beta_i$

So $\alpha_i(t)\cdot\alpha_i'(t)=0$, $\alpha_i(t)\alpha_j'(t)+\alpha_i'(t)\alpha_j(t)=0$

$\Downarrow$ i^{th} element of $\alpha_i'(t)=0$, $\Downarrow$ i^{th} element of $\alpha_j'(t)+j^{th}$ element of $\alpha_i'(t)=0$

which yields the same result/conclusion.

15.9 $v\in S_p=\{v\in R_p^{n+1}\mid \nabla f_i(p)\cdot v=0\}\Leftrightarrow \nabla f_i(p)\cdot v=0\ \forall i \Leftrightarrow v\in \operatorname{Ker} df_p$

15.10 (brief proof). Since $J = \begin{pmatrix}\nabla f_1\\ \vdots \\ \nabla f_k\end{pmatrix}$ is fully ranked, so there are k independent columns indexed by $i_1 \cdots i_k$, which form matrix P. So $\det P \neq 0$. Define $\Psi(x_1 \cdots x_n)$ as $\Psi(x_1 \cdots x_{i_1-1}, f_1(x_1 \cdots x_{n+1}), x_{i_1+1}, \cdots, x_{i_k-1}, f_k(x_1 \cdots x_n), x_{i_k+1}, \cdots x_{n+1})$, whose Jacobian J satisfies $\det(J) = \det(P) \neq 0$. Then go on as in proof of Thm 1 by applying inverse function theorem.
Finally, $U = \{(u_1 \cdots u_n) \in R^n \mid a_j < u_j < b_j$ for $j < i_1$, $a_{j+1} < u_j < b_{j+1}$ for $j \in [i_1, i_2)$, $\cdots$, $a_{j+k-1} < u_j < b_{j+k-1}$ for $j \in [i_{k-1}, i_k)$, $a_{j+k} < u_j < b_{j+k}$ for $j \geq i_k\}$ and define $\varphi: U \to R^{n+k}$ by
$\varphi(u_1 \cdots u_n) = (\Psi|_V)^{-1}(u_1 \cdots u_{i_1-1}, c_1, u_{i_1+1}, \cdots, u_{i_k-1}, c_k, u_{i_k+1}, \cdots, u_n)$. (elsewhere, just change $n+1$ to $n+k$ in proof of Thm 1)

15.11 (brief proof). Define $U \times R^k \to R^{n+k}$ by $\psi(q, t_1 \cdots t_k) = \varphi(q) + \sum_{i=1}^{k} t_i N_i(q)$, where N_i are the vector fields along φ which span the normal space $(\text{Image } d\varphi_q)^\perp$ for each $q \in U$. Then Jacobian $J\psi(p, 0 \cdots 0) = (J\varphi(p), N_1(p) \cdots N_k(p))$ whose determinant $\neq 0$. By the inverse func Thm, there is an open set $V \subset U \times R^k$ about $(p, \overbrace{0 \cdots 0}^{k\ 0's})$ such that the restriction $\psi|_V$ of ψ to V maps V one to one onto the open set $\psi(V)$, and $(\psi|_V)^{-1}$ is smooth. By shrinking V if necessary, we may assume $V = U_1 \times I^k$ for some open set $U_1 \subseteq U$ containing p and some interval $I \subseteq R$ containing 0. Now define f: Image $\psi|_V \to R^k$ by $f(\psi(q, t_1 \cdots t_k)) = (t_1, \cdots, t_k)$. f is well defined and is smooth because f is the composition of the smooth map $(\psi|_V)^{-1}$ and projection map $U_1 \times I^k \to I^k$. The level set $f^{-1}(0 \cdots 0)$ is just $\varphi(U_1)$ because $f^{-1}(0) = \{\psi(q, t_1 \cdots t_k) \mid q \in U_1, t_i = 0\} = \{\varphi(q) \mid q \in U_1\}$. Finally we prove that $Jf(z)$ is fully ranked for $z = \psi(q, 0 \cdots 0) \in f^{-1}(0 \cdots 0)$. Let $\alpha_i(s) = \varphi(q) + s \cdot N_i(q)$ then $\nabla f_j(z) \cdot N_i(q)$
$= \nabla f_j(\alpha_i(0)) \cdot \dot{\alpha}_i(0) = \cancel{f_j} (f_j \circ \alpha_i)^{\cdot}(0) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. So

$\underbrace{\begin{pmatrix}\nabla f_1(z)\\ \vdots \\ \nabla f_k(z)\end{pmatrix}}_{\triangleq A} \cdot \underbrace{(N_1(q) \cdots N_k(q))}_{\triangleq B} = I_k$. ~~By definition rank $(N_1(q) \cdots N_k(q)) = k$~~ To be fast, let's quote a matrix result: $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

Now $k \leq \min(\text{rank}(A), \text{rank}(B))$. But both $\text{rank}(A)$ and $\text{rank}(B) \leq k$ thus
$\text{rank}(A) = \text{rank}(B) = k$, i.e., A (Jacobian) is fully ranked.
To prove $\text{rank } Jf(z) = k$, another way is: assume $\sum_{i=1}^{k} \beta_i \nabla f_i(z) = 0$, then $\sum_{i=1}^{k} \beta_i (\nabla f_i(z) B) = 0$.
But $\nabla f_i(z) \cdot B$ is just the i^{th} row of I_k. So $\beta_i = 0$ for all $i = 1 \cdots k$, i.e. $\{\nabla f_1(z), \cdots, \nabla f_k(z)\}$ are independent. Thus $\varphi(U_1) = f^{-1}(0 \cdots 0)$ is an n-surface in R^{n+k}

~~15.12~~

15.12 (a) $\varphi(p+tv) = \left(2(x_1+tv_1), \cdots, 2(x_n+tv_n), \sum_{i=1}^{n}(x_i+tv_i)^2 - 1\right) / \left(1 + \sum_{i=1}^{n}(x_i+tv_i)^2\right)$

$\frac{d}{dt}\Big|_0 \varphi(p+tv) = \left(2v_i\left(\sum_{j=1}^{n} x_j^2 + 1\right) - 4x_i \sum_{j=1}^{n} x_j v_j \text{ for } i = 1 \cdots n,\ -4 \sum_{i=1}^{n} x_i v_i\right) / \left(\sum_{j=1}^{n} x_j^2 + 1\right)^2$

So $\|d\varphi(v)\|^2 = \|\frac{d}{dt}|_0 \varphi(p+tv)\|^2 = 4\left\{\sum_{i=1}^{n}\left[v_i\left(\sum_{j=1}^{n} x_j^2 + 1\right) - 2x_i \sum_{j=1}^{n} x_j v_j\right]^2 + 4\left(\sum_{i=1}^{n} x_i v_i\right)^2\right\} / \left(\sum_{j=1}^{n} x_j^2 + 1\right)^4$

$= 4\left(\sum_{j=1}^{n} x_j^2 + 1\right)^{-2} \|v\|^2$ So $\lambda(p) = \frac{2}{\|p\|^2 + 1}$

(b) $d\varphi(v) \cdot d\varphi(w) = \frac{1}{4}\left(\|d\varphi(v) + d\varphi(w)\|^2 - \|d\varphi(v) - d\varphi(w)\|^2\right)$ then by linearity of $d\varphi_p$

$= \frac{1}{4}\left(\|d\varphi(v+w)\|^2 - \|d\varphi(v-w)\|^2\right) = \frac{1}{4}\lambda^2(p)\left(\|v+w\|^2 - \|v-w\|^2\right) = \lambda^2(p)\cdot v\cdot w.$

15.13 Let $\tilde{S} = \{q \in S \mid q$ can be joined to p by a continuous curve in $S\}$. Let $S = f^{-1}(c)$
First $\tilde{S}$ is obviously connected. $\forall q_1, q_2 \in \tilde{S}$, just concatenate their curve joining p will yield a continuous curve between q_1 and q_2. Since $\tilde{S} \subseteq S$, so $\forall q \in \tilde{S}$, $\nabla f(q) \neq 0$. Now we only need to prove that there is an open set ~~s.t.~~ U, s.t. ~~$\tilde{S} = \{x \in U \mid f(x) = c\}$~~. $\tilde{S} = \{x \in U \mid f(x) = c\}$. We mimic the proof of Thm 3
For each $q \in \tilde{S} \subseteq S$, let $\varphi_q : U_q \to S$ be a local parametrization of S whose image contains q and let $\psi_q : U_q \times R \to R^{n+1}$ be defined by $\psi_q(r,s) = \varphi_q(r) + s N(\varphi_q(r))$, where N is the orientation of S. Then as in the proof of Thm 2, we can find an open set V_q about $(\varphi_q^{-1}(q), 0)$ in $U_q \times R$ s.t. $\psi_q|_{V_q}$ maps V_q one to one onto and open set W_q in R^{n+1}, and $(\psi_q|_{V_q})^{-1} : W_q \to V_q$ is smooth. Furthermore by shrinking V_q if neccssary, we may assume that $\psi_q(r,s) \in S$ for $(r,s) \in V_q$ iff $s=0$. ~~Since V_q is an open set,~~ then for any $u \in W_q \cap S$, let $(\psi_q|_{V_q})^{-1}(u) = (r,0)$ must be ~~there must be a unique~~ $r \in U_q'$ ~~s.t. $\psi_q(r,0) = u$~~. Since U_q' is open and connected, there is a smooth curve $\alpha(t) : [a,b] \to U_q'$ s.t. ~~$\alpha(t) \in V_q$~~, $\alpha(a) = \varphi_q^{-1}(q)$, $\alpha(b) = r$, (actually we should define $\alpha(t)$ in an open set containing $[a,b]$). Since ~~$\alpha(t) \in V_q$~~

By shrinking V_q further, we may assume that $V_q = U_q' \times I$ where $U_q' \subseteq U_q$, $I \subseteq R$, $0 \in I$, $\varphi_q^{-1}(q) \in U_q'$, U_q' open, I open, U_q' connected

As $\alpha(t) \in U_q'$ and $0 \in I$, we have $\beta(t) \triangleq \psi_q(\alpha(t); 0) \in W_q \cap S$. So $\beta(b) = u$ is connected to $\beta(a) = q$ through a continuous curve on S, so $u \in \tilde{S}$. In other words, for $\forall q \in \tilde{S}$, there is an open set W_q about q, s.t. $W_q \cap S \subseteq \tilde{S}$.
Now we define $U = \bigcup_{q \in \tilde{S}} W_q$ which is open, ~~then $\tilde{S} \subseteq U$ by definition~~.
① $\forall x \in \tilde{S}$, we have $x \in U$, $f(x) = c$. So $x \in \{x \in U \mid f(x) = c\}$, so $\tilde{S} \subseteq \{x \in U \mid f(x) = c\}$
② $\forall x \in \{x \in U \mid f(x) = c\}$, there must be a $q \in \tilde{S}$, s.t. $x \in W_q$. As $x \in S$, so $x \in W_q \cap S \subseteq \tilde{S}$. Thus, $\{x \in U \mid f(x) = c\} \subseteq \tilde{S}$
Hence $\tilde{S} = \{x \in U \mid f(x) = c\}$, i.e. $\tilde{S}$ is n-surface.

15.14 Suppose $\alpha(t_1) = \alpha(t_2)$ for some $t_1 \neq t_2 \in I$. As the maximal integral curve of X through $\alpha(t_1)$ is unique, denoted as $\beta(t)$ and $\beta(0) = \alpha(t_1)$, then $\alpha(t) = \beta(t - t_1)$ and $\alpha(t) = \beta(t - t_2)$ for all $t \in I$. Setting $\tau = t_2 - t_1$, we have $\alpha(t) = \beta(t - t_1) = \beta(t + \tau - t_2) = \alpha(t + \tau)$ for all t such that both t and $t + \tau \in I$
Thus if α is not one to one then it is periodic.
To prove that the maximal integral curve X through $\alpha(t_1)$ is ~~unique~~ constrained in C, We notice

that first the restriction of X to C is a tangent vector field on C, because $\overset{\text{for } i=1,2}{X \cdot \nabla f_i} = (\nabla f_1 \times \nabla f_2) \cdot \nabla f_i = 0$

So $(f_i \circ \alpha)'(t) = \nabla f_i(\alpha(t)) \cdot \dot{\alpha}(t) = \nabla f_i(\alpha(t)) \cdot X(\alpha(t)) = 0$. So $f_i \circ \alpha$ is constant, thus $\alpha(t) \in C$.

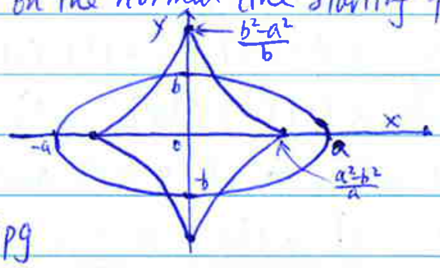
~~To make~~ α map I onto C, ~~first of all,~~ iff C ~~must be~~ is connected. The proof is similar to the Thm 1 in Chapter 11, except the construction of rectangle B, we now construct $A = \{ p_0 + s_1 v_1 + s_2 v_2 + ru \mid |s_i| < \varepsilon_i, |r| < \varepsilon_3 \}$ where $v_i = \nabla f_i(p_0)$, $u = X(p_0)$. The $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are chosen ~~as follows. First~~, so that $J\vec{f}(p)$ is fully ranked for all $p \in A$. This is possible because $J\vec{F}(p_0)$ is fully ranked. Denote $g_r(s_1, s_2) = \vec{f}(p_0 + s_1 v_1 + s_2 v_2 + ru)$, then $Jg_r = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix} (v_1, v_2) = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix} (\nabla f_1, \nabla f_2)$. As $\operatorname{rank}(P'P) = \operatorname{rank}(P) = \operatorname{rank}(P')$, $\operatorname{rank}\begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix}_p = 2$ for $\forall p \in A$. So Jg_r is fully ranked for all $p \in A$. Applying Lagrange Mean Value Thm ~~Inverse function Thm~~, if r is chosen such that $\exists\, s_1, s_2$ s.t. $\vec{f}(p_0 + s_1 v_1 + s_2 v_2 + ru) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, then such s_1, s_2 are unique in $(-\varepsilon_1, \varepsilon_1) \times (-\varepsilon_2, \varepsilon_2)$.

Now let $\gamma(t) = p_0 + h_0(t) u + h_1(t) v_1 + h_2(t) v_2$, where $h_0(t) = (\gamma(t) - p_0) \cdot u / \|u\|^2$,
$h_i(t) = (\gamma(t) - p_0) \cdot v_i / \|v_i\|^2$ $(i = 1, 2)$. h_1, h_2, h_0 are all smooth and $h_1(0) = h_2(0) = h_0(0) = 0$.
$h_0'(0) = \dot{\gamma}(0) \cdot u / \|u\|^2 = 1$ by definition that $\dot{\gamma}(0) = X(p_0) = u$. So we can choose $t_1 < 0 < t_2$ (small enough), s.t. $h_0'(t) > 0$, set $r_1 = h_0(t_1)$, $r_2 = h_0(t_2)$, then for $\forall r \in (r_1, r_2)$, $\exists!\, t \in (t_1, t_2)$, s.t. $h_0(t) = r$. Now construct $B = \{ p_0 + ru + s_1 v_1 + s_2 v_2 \mid r_1 < r < r_2, |s_i| < \varepsilon_i \}$ $(i = 1, 2)$.
$\forall\, p_0 + ru + s_1 v_1 + s_2 v_2 \in B \cap C$, then its belonging to $B \Rightarrow \exists!\, t \in (t_1, t_2)$, $h_0(t) = r$.
Its belonging to $C \Rightarrow \exists!\, s_1, s_2$ s.t. $p_0 + ru + s_1 v_1 + s_2 v_2 \in C$. Now that we know $p_0 + h_0(t) u + h_1(t) v_1 + h_2(t) v_2 \in \gamma \subseteq C$, so $s_1 = h_1(t)$, $s_2 = h_2(t)$. So $p_0 + ru + s_1 v_1 + s_2 v_2 \in \gamma$.
i.e. $C \cap B \subseteq \gamma$

16.1 (a) By using the rest of Ex 10.4 (b) at $p = (a\cos t, b\sin t)$, the curvature for outward orientation is $k(p) = -ab\left(\frac{a^2}{b^2} x_2^2 + \frac{b^2}{a^2} x_1^2\right)^{-\frac{3}{2}}$, $N = (a^2 \sin^2 t + b^2 \cos^2 t)^{-\frac{1}{2}} (b\cos t, a\sin t)$.

By applying Thm 1, the focal point on the normal line starting from p is $p + \frac{1}{k(p)} N(p)$
$= \left(\frac{a^2 - b^2}{a} \cos^3 t,\ \frac{b^2 - a^2}{b} \sin^3 t\right)$

(b) For example of $a = 2$, $b = 1$, see
http://rsise.anu.edu.au/~xzhang/reading/ex1601.jpg



16.2 (a) Only need to prove that for q sufficiently close to p, $N(p)$ and $N(q)$ are not parallel in $\mathbb{R}^2$.
Otherwise, for $\forall k \in \mathbb{Z}^+$, $\exists\, q_k \in C \cap \varepsilon(p, \varepsilon)$ (the ε-ball about p), such that $N(p)$ and $N(q_k)$ are parallel. But as N is smooth, $\boxed{\text{So } \|N(p) - N(q_k)\| = 0 \text{ or } 2}$, in the neighbourhood close enough to p, $\|N(p) - N(q_k)\|$ must be less than an arbitrary small positive number.
So $N(q_k) = N(p)$, so $\underbrace{\cancel{(N(q_k) - N(p))}}_{JN(p + \lambda_k(q_k - p))} \cdot \frac{(q_k - p)}{\|q_k - p\|} = 0$ $(\lambda_k \in (0,1))$. As $(q_k - p)/\|q_k - p\| \in S^1$ which is compact,

there must be a subsequence of $(q_k - p)/\|q_k - p\|$ which converges to v ($\|v\|=1$). Without loss of generality, we assume that subsequence is $\{q_k\}$ itself. Let $k \to \infty$, We have $\nabla_v N = 0$, because $\lim_{k\to\infty} \frac{q_k - p}{\|q_k - p\|} = v$. $\lim_{k\to\infty} p + \lambda_k (q_k - p) = p$ as $q_k \to p$. $\nabla_v N = 0$ contradicts with the assumption that the curvature $k(p) \neq 0$. So for $q \in C$ sufficiently close to p, the normal lines to C at p and q intersects at some point $h(q) \in \mathbb{R}^2$

(b) First derive $h(q)$. $p + s_1 \cdot N(p) = q + s_2 \cdot N(q)$. Suppose there is a local parametrization of C about p: $\alpha(t): I \to C$, $\alpha(t_0) = p$, and suppose I is small enough s.t. $\forall t \in I$, $\alpha(t)$ satisfies (a). So to derive $h(\alpha(t))$, suppose $\alpha(t) + s_2 N(\alpha(t)) = \alpha(t_0) + s_1 N(\alpha(t_0))$ ($s_1, s_2 \in \mathbb{R}$). Multiply both sides by $\dot\alpha(t_0)$ and Notice $N(\alpha(t_0)) \cdot \dot\alpha(t_0) = 0$, so

$\alpha(t) \cdot \dot\alpha(t_0) = \cancel{\alpha(t_0)\dot\alpha(t)} + s_2 N(\alpha(t)) \cdot \dot\alpha(t_0) = \alpha(t_0)\dot\alpha(t_0)$. By assumption $N(\alpha(t))$ is not parallel with $\alpha(t_0)$, so $N(\alpha(t)) \cdot \alpha(t_0) \neq 0$ So $s_2 = \frac{\alpha(t_0)\cdot\dot\alpha(t_0) - \alpha(t)\dot\alpha(t_0)}{N(\alpha(t)) \cdot \dot\alpha(t_0)}$

So $h(\alpha(t)) = \frac{1}{N(\alpha(t)) \cdot \dot\alpha(t_0)} \left[\alpha(t) \cdot \left(N(\alpha(t)) \cdot \dot\alpha(t_0)\right) + N(\alpha(t)) \cdot \left(\alpha(t_0)\dot\alpha(t_0) - \dot\alpha(t_0)\alpha(t)\right)\right]$

Both numerator and denominator $\to 0$ as $t \to t_0$. So using L'Hospital's rule, the derivative of denominator is $\dot{N\circ\alpha}(t) \cdot \dot\alpha(t_0)$ which equals $-k(t_0)\|\dot\alpha(t_0)\|^2$

the derivative of numerator is $\dot\alpha(t) \cdot \left(N(\alpha(t)) \cdot \dot\alpha(t_0)\right) + \alpha(t)\left[\dot{N\circ\alpha}(t) \cdot \dot\alpha(t_0)\right]$

$+ \dot{N\circ\alpha}(t) \cdot \left(\alpha(t_0) \cdot \dot\alpha(t_0)\right) - \dot{N\circ\alpha}(t_0) \cdot \left[\dot\alpha(t_0) \cdot \alpha(t)\right] - N(\alpha(t)) \cdot \left[\dot\alpha(t_0) \cdot \dot\alpha(t)\right]$

when $t = t_0$, it is equal to $\alpha(t_0) \cdot \left(-k(t_0)\|\dot\alpha(t_0)\|^2\right) - k(t_0)\dot\alpha(t_0)\left(\alpha(t_0) \cdot \dot\alpha(t_0)\right)$

$+ k(t_0)\dot\alpha(t_0)\left[\dot\alpha(t_0) \cdot \alpha(t_0)\right] - N(\alpha(t_0)) \cdot \|\dot\alpha(t_0)\|^2$

So $\lim_{t\to t_0} h(\alpha(t)) = -\|\dot\alpha(t_0)\|^2 \left(N\circ\alpha(t_0) + k(t_0) \cdot \alpha(t_0)\right) / \left[-k(t_0)\|\dot\alpha(t_0)\|^2\right]$

$= \alpha(t_0) + \frac{1}{k(t_0)}(N\circ\alpha)(t_0)$ $\quad (\alpha(t_0) = p)$

By Thm 1, this is the focal point of C along the normal line through p.

16.3 (a) $\dot\alpha(t) = \varphi'(t) + \frac{1}{k^2(t)}\left[(\dot{N\circ\varphi})(t)\, k(t) - k'(t)\,(N\circ\varphi)(t)\right]$

As $(\dot{N\circ\varphi})(t) = -L_p(\dot\varphi(t)) = -k(t)\,\dot\varphi(t)$.

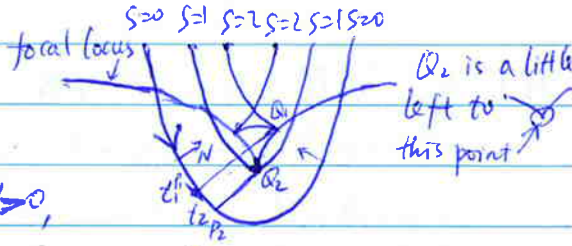
So $\dot\alpha(t) = \frac{-1}{k^2(t)} k'(t)\,(N\circ\varphi)(t)$. So $\dot\alpha(t) = 0$ iff $k'(t) = 0$

(b) As $\dot\alpha(t)$ is parallel to $(N\circ\varphi)(t)$ and by definition $\alpha(t)$ is on the normal line to Image φ at $\varphi(t)$ and by Thm 1, $\alpha(t)$ is the focal locus of φ. So the latter is tangent ~~at~~ $\alpha(t)$ to the focal locus of φ

(c) The sum is $Q = \int_{t}^{t_2} \|\dot\alpha(t_1)\|\, dt_1 + \|\alpha(t) - \varphi(t)\|$. Suppose $k'(t) = b\|k'(t)\|$ and $k(t) = a\|k(t)\|$ where $a, b \in \{\pm 1\}$ as both $k(t)$ and $k'(t)$ keep their sign for $t \in (t_1, t_2)$. So $\frac{d}{dt} Q = -\|\dot\alpha(t)\| + \frac{d}{dt}\left|\frac{1}{k(t)}\right| = -\|\dot\alpha(t)\| + a\frac{d}{dt}\frac{1}{k(t)}$

$= \frac{-1}{k^2(t)}\, b\, k'(t) + a\frac{-1}{k^2(t)} k'(t)$. Notice that if $a \cdot b = 1$ then the conclusion in this exercise doesn't hold. Otherwise if $k'(t) \cdot k(t) < 0$. $\frac{d}{dt} Q = 0$ so $Q =$ constant

An example of $ab=1$ is the ~~fo~~ parabola. If we parametrize by $\varphi(t) = (t, \frac{1}{2}t^2)$ $t\in(-\infty, 0)$
then $\dot\varphi(t) = (1,t)$, $N=\frac{1}{\sqrt{t^2+1}}(-t,+1)$, $k(t) = (1+t^2)^{-3/2}>0$, $k'>0$,
then $|P_1Q_1| + |\text{length of } \alpha\ \alpha(t_1)\to\alpha(t_2)| > |P_2Q_2| + 0$. can't be constant.
So we will need $kk'<0$, like what the Figure 16.6 shows



16.4 (a) Let $\psi(s,t) = \varphi(t) + sN(\varphi(t))$. For each $s<\frac{1}{k(t_0)}$, $\varphi_s(t_0)$ is not a focal point by Thm 1, so $I_s\neq\phi$. If $t_0\in I_s$, then $\varphi_s'(t_0)\neq 0$. As φ_s' is continuous, there must be $\varepsilon>0$ s.t. $\forall t\in(t_0-\varepsilon, t_0+\varepsilon)$, $\varphi_s'(t)\neq0$, i.e. $t\in I_s$. Thus I_s is open.

(b) Suppose $\varphi(t)$ is unit speed, which doesn't lose generality as the conclusion only takes care of t_0. $\varphi_s(t) = \varphi(t)+sN(\varphi(t))$, $k(t_0) = \ddot\varphi(t_0)\cdot N(\varphi(t_0))$

$\varphi_s'(t) = \varphi'(t) + s(\dot{N\circ\varphi})(t)$ As $\varphi_s'(t)\cdot(N\circ\varphi)(t) = (\varphi'(t)+s(\dot{N\circ\varphi})(t))(N\circ\varphi)(t)$

by definition $\varphi'(t)\cdot(N\circ\varphi)(t)=0$. $\|(N\circ\varphi)(t)\| = \text{const} \Rightarrow (\dot{N\circ\varphi})(N\circ\varphi)=0$, so $\varphi_s'(t)\cdot(N\circ\varphi)(t)=0$

~~So $N_s(\varphi_s(t)) = N(\varphi(t))$~~. To check the direction, we notice that

$\varphi_s'(t)\cdot\varphi'(t) = (\varphi'(t) + s\cdot(\dot{N\circ\varphi})(t))\cdot\varphi'(t) = \|\varphi'(t)\|^2 + s(-k\|\varphi'(t)\|^2)$

As $s<\frac{1}{k(t_0)}$. So if $k(t_0)>0$, then $\varphi_s'(t)$ is in the same direction as $\varphi'(t)$, and $N_s(\varphi_s(t)) = N(\varphi(t))$. If $k(t_0)<0$, then $N_s(\varphi_s(t)) = -N(\varphi(t))$.

1°. $k(t_0)>0$, $k_s(t_0) = \ddot\varphi_s(t_0) N_s(\varphi_s(t_0))/\|\varphi_s'(t_0)\|^2$

$= (\ddot\varphi(t_0) + s(\ddot{N\circ\varphi})(t_0)) N(\varphi(t_0))/\|\varphi_s'(t_0)\|^2$

$\ddot\varphi(t_0)\cdot N(\varphi(t_0)) = k(t_0)$. Besides, as $-(\dot{N\circ\varphi}) = k\cdot\dot\varphi$,

so $-(\ddot{N\circ\varphi}) = k'\varphi' + k\cdot\ddot\varphi$, so $-(\ddot{N\circ\varphi})(N\circ\varphi) = k\cdot\ddot\varphi(N\circ\varphi) = +k^2$

So $k_s(t_0) = (k(t_0) - sk^2(t_0))/\|\varphi_s'(t_0)\|^2$

$\varphi_s'(t_0) = \varphi'(t_0)+s(\dot{N\circ\varphi})(t_0) = \varphi'(t_0) + s\cdot(-k(t_0)\varphi'(t_0))$. So $\|\varphi_s'(t_0)\| = |1-sk(t_0)|$

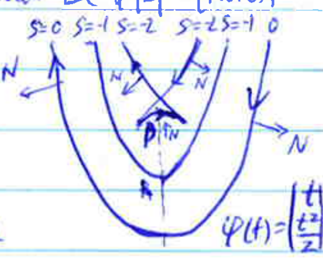
So $k_s(t_0) = (k(t_0)-sk^2(t_0))/(1-sk(t_0))^2 = (\frac{1}{k(t_0)}-s)^{-1}$

2° $k(t_0)<0$. $k_s(t_0) = -\ddot\varphi_s(t_0) N(\varphi(t_0))/\|\varphi_s'(t_0)\|$ similar to above, we have

~~$\ddot\varphi_s(t_0)$~~ $= k_s(t_0) = -(\frac{1}{k(t_0)}-s)^{-1}$ So we ~~ϕ~~ suspect that it should be ~~ϕ~~ $|s|<|\frac{1}{k(t_0)}|$

or simply assume $k(t_0)>0$. To double check we are correct, see parabola again and $\varphi(t_0)=(0,0)$, $y=\frac{1}{2}x^2$. $k(0)=-1$, $\varphi(t)=\binom{t}{\frac{1}{2}t^2}$

Let $s=-2<1/k(0)$. at P, the curvature should still be negative, while the conclusion in the textbook exercise insists $k_{-2}(0) = \frac{1}{-1-(-2)} = 1>0$



16.5 (a) $J\psi|_{t=0} = \left(N^S(\alpha(0)), \dot\alpha(0) + s(\dot{N}\circ\alpha)(0)\right)$

so $X(S) = J\psi|_{t=0}\cdot\begin{pmatrix}0\\1\end{pmatrix} = \dot\alpha(0) + S(\dot{N}\circ\alpha)(0)$

$X(0) = \dot\alpha(0) = v$. $\dot X(S) = (\dot N\circ\alpha)(0)$ So $\dot X(0) = (\dot N\circ\alpha)(0) = -L_p(v)$

(b) $\ddot X(S) = 0$. So $X(S) = \left(\psi(s,0), \dot\alpha(0) + s(\dot N\circ\alpha)(0)\right) = (\beta(S), v + sw)$

(c) $X(S) = 0 \Leftrightarrow v = -s\cdot(\dot N\circ\alpha)(0) = sL_p(v)$

So $\frac{1}{s}$ is a principle curvature and v is a principal curvature direction

By Thm 1, $\Leftrightarrow \alpha(0) + \frac{1}{\frac{1}{s}}\cdot N^S(\alpha(t)) = \alpha(0) + sN^S(\alpha(t)) = \beta(S)$ is focal point of S along β.

17.1 $V(\varphi) = \int_0^{2\pi}\int_0^h \left|\det\begin{vmatrix}-r\sin\theta & r\cos\theta & 0\\ 0 & 0 & 1\\ \cos\theta & \sin\theta & 0\end{vmatrix}\right| dt\,d\theta = \int_0^{2\pi}\int_0^h r\,dt\,d\theta = 2\pi h$

17.2 $E_1 = \frac{\partial\varphi}{\partial\theta} = (-tr\sin\theta, tr\cos\theta, 0)$, $E_2 = \frac{\partial\varphi}{\partial t} = (r\cos\theta, r\sin\theta, -h)$

$E_1\cdot E_1 = t^2r^2$. $E_2\cdot E_2 = r^2+h^2$ $E_1\cdot E_2 = 0$ $V(\varphi) = \int_0^{2\pi}\int_0^1 \sqrt{t^2r^2(r^2+h^2)}\,dt\,d\theta = \pi r\sqrt{r^2+h^2}$

17.3 $E_1 = \frac{\partial\varphi}{\partial\theta} = (-(a+b\cos\phi)\sin\theta, (a+b\cos\phi)\cos\theta, 0)$, $E_2 = \frac{\partial\varphi}{\partial\phi} = (-b\sin\phi\cos\theta, -b\sin\phi\sin\theta, +b\cos\phi)$

$E_1\cdot E_1 = (a+b\cos\phi)^2$, $E_2\cdot E_2 = b^2$, $E_1\cdot E_2 = 0$ $V(\varphi) = \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos\phi)\,d\phi\,d\theta = 4\pi^2 ab$

17.4 $E_1 = \frac{\partial\varphi}{\partial\theta} = (-a\sin\theta, a\cos\theta, 0, 0)$, $E_2 = \frac{\partial\varphi}{\partial\phi} = (0, 0, -b\sin\phi, b\cos\phi)$, $E_1\cdot E_1 = a^2$, $E_2\cdot E_2 = b^2$, $E_1\cdot E_2 = 0$

$V(\varphi) = \int_0^{2\pi}\int_0^{2\pi} ab\,d\theta\,d\phi = 4\pi^2 ab$

17.5 $E_1 = \frac{\partial\varphi}{\partial\phi} = (\cos\phi\sin\theta\sin\psi, -\sin\phi\sin\theta\sin\psi, 0, 0)$ $E_2 = \frac{\partial\varphi}{\partial\theta} = (\sin\phi\cos\theta\sin\psi, \cos\phi\cos\theta\sin\psi, -\sin\theta\sin\psi, 0)$

$E_3 = \frac{\partial\varphi}{\partial\psi} = (\sin\phi\sin\theta\cos\psi, \cos\phi\sin\theta\cos\psi, \cos\theta\cos\psi, -\sin\psi)$.

$E_1\cdot E_1 = (\sin\theta\sin\psi)^2$, $E_2\cdot E_2 = (\sin\psi)^2$, $E_3\cdot E_3 = 1$. $E_i\cdot E_j = 0$ $(i\neq j)$.

So $V(\varphi) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} \sin\theta(\sin\psi)^2\,d\phi\,d\theta\,d\psi = 2\pi^2$

17.6 $E_1 = (x'(t), y'(t)\cos\theta, y'(t)\sin\theta)$, $E_2 = (0, -y(t)\sin\theta, y(t)\cos\theta)$

$E_1\cdot E_1 = x'(t)^2 + y'(t)^2$, $E_2\cdot E_2 = y(t)^2$, $E_1\cdot E_2 = 0$.

$V(\varphi) = \int_a^b\int_0^{2\pi}\sqrt{x'(t)^2+y'(t)^2}\,y(t)\,d\theta\,dt = 2\pi\int_a^b y(t)\left(x'(t)^2+y'(t)^2\right)^{1/2}dt$

17.7 Let $a_i = \frac{\partial g}{\partial u_i}$ then $E_i = \frac{\partial\varphi}{\partial u_i} = (0,\cdots 0, 1, 0, \cdots, 0, a_i)$ $N = \pm(a_1, \cdots, a_n, -1)/\left(1+\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$

As $\begin{vmatrix}1 & & & a_1\\ & \ddots & & \vdots\\ & & 1 & a_n\\ a_1 & \cdots & a_n & -1\end{vmatrix} = \cancel{\begin{vmatrix}1+a_1^2 & & 0\\ & \ddots & \\ a_1 & \cdots\ 1+a_n^2 & 0\\ & a_n & 1\end{vmatrix} = \prod_{i=1}^n(1+a_i^2)}$, So $= \begin{vmatrix}1 & & 0\\ & \ddots & \\ & & 1\ \ a_n\\ a_1 & & a_n\ \ -1-a_1^2\end{vmatrix} = \begin{vmatrix}1 & & a_2\\ & \ddots & \vdots\\ & & 1\ \ a_n\\ a_2 & \cdots\ a_n & -1-a_1^2\end{vmatrix} = \cdots = -\left(1+\sum_{i=1}^n a_i^2\right)$

So $V(\varphi) = \int_U \left(1+\sum_{i=1}^{n} a_i^2\right) \Big/ \left(1+\sum_{i=1}^{n} a_i^2\right)^{1/2} = \int_U \left(1+\sum_{i=1}^{n} (\partial g/\partial u_i)^2\right)^{1/2}$.

17.8 (a) Prove $J\varphi_n$ is not singular. We prove by induction. When $n=2$. $J_2 = \begin{pmatrix} \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 & 0 \\ \sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_2 \end{pmatrix}$

$\operatorname{rank} J_2 = \operatorname{rank} J_2 \cdot J_2^T = \operatorname{rank}\begin{pmatrix} \sin^2\theta_2 & 0 \\ 0 & 1 \end{pmatrix} = 2$, So J_2 is fully ranked. Suppose $\operatorname{rank} J_{n-1} = n-1$.

Denoted the $(i,j)^{th}$ component of J_n as $J_n^{i,j}$, $i=1\cdots n+1$, $j=1\cdots n$. then

$$J_n = \begin{pmatrix} J_{n-1}^{1,1}\sin\theta_n & \cdots & J_{n-1}^{1,n-1}\sin\theta_n & \varphi_{n-1}^{1}\cos\theta_n \\ & \ddots & \vdots & \vdots \\ & & J_{n-1}^{n,n-1}\sin\theta_n & \varphi_{n-1}^{n}\cos\theta_n \\ 0 & \cdots & 0 & -\sin\theta_n \end{pmatrix} = \begin{pmatrix} \sin\theta_n \cdot J_{n-1} & \varphi_{n-1}\cos\theta_n \\ 0 & -\sin\theta_n \end{pmatrix}$$

, so $\operatorname{rank} J_n = n-1+1 = n$. i.e., φ_n is parametrized n-surface.

(b) ~~φ_n maps U one to one onto a subset of unit n-sphere S^n: $\{(a_1\cdots a_{n+1}) \mid \sum_{i=1}^{n+1} a_i = 1\}$~~,

Let φ_n^i be the i^{th} component of φ_n. then $\sum_{i=1}^{n+1} {\varphi_n^i}^2 = 1$ So φ_n maps U to a subset of S^n.

We only need to prove one to one. If $\varphi_n(\theta_1\cdots\theta_n) = \varphi_n(\hat\theta_1,\cdots,\hat\theta_n)$, then

$\cos\theta_n = \cos\hat\theta_n$, As $\theta_n, \hat\theta_n \in (0,\pi)$. so $\theta_n = \hat\theta_n$. as $\sin\hat\theta_n \neq 0$ So

$\varphi_{n-1}(\theta_1,\cdots,\theta_{n-1}) = \varphi_{n-1}(\hat\theta_1,\cdots,\hat\theta_{n-1})$. For the same reason, we have $\theta_{n-1} = \hat\theta_{n-1}, \cdots, \theta_2 = \hat\theta_2$.

Finally $(\sin\theta_1, \cos\theta_1) = (\sin\hat\theta_1, \cos\hat\theta_1)$. As $\theta_1, \hat\theta_1 \in (0, 2\pi)$, $\theta_1 = \hat\theta_1$. Thus φ_n is one to one

(c) ~~If $x \in S^n - \text{Image}\,\varphi_n$, then $\prod_{i=1}^{n}\sin\theta_i = 0$. This is because if $\prod_{i=1}^{n}\sin\theta_i \neq 0$,~~

~~$\varphi(\theta_1,\cdots,\theta_n)$~~ It is obvious that $\hat\varphi_n : U' \to R^{n+1}$ with $U' = \{(\theta_1\cdots\theta_n) \in R^n \mid 0 \le \theta_1 < 2\pi, 0 \le \theta_i < \pi\ i \in [2,n]\}$

maps onto S^n. So if $x = \hat\varphi(\theta_1\cdots\theta_n) \in S^n - \text{Image}\,\varphi$, then $(\theta_1\cdots\theta_n) \in U' \setminus U$.

so $\prod_{i=1}^{n}\sin\theta_i = 0$ So $x_1 = 0$. Thus $S^n - \text{Image}\,\varphi$ is contained in the $(n-1)$-sphere

$\{(x_1\cdots x_{n+1}) \in S^n \mid x_1 = 0\}$. So $V(\varphi_n) = V(S^n)$

(d) $|\xi_n| = \left|\begin{pmatrix} \sin\theta_n \cdot J_{n-1} & \varphi_{n-1}\cos\theta_n & \varphi_{n-1}\sin\theta_n \\ 0 & -\sin\theta_n & \cos\theta_n \end{pmatrix}\right| = \left|\begin{pmatrix} \sin\theta_n J_{n-1} & \varphi_{n-1}\cos\theta_n & \varphi_{n-1}\frac{1}{\sin\theta_n} \\ 0 & -\sin\theta_n & 0 \end{pmatrix}\right| = (\sin\theta_n)^n |J_{n-1}\ \varphi_{n-1}|$
$= (\sin\theta_n)^n |\xi_{n-1}|$

So $V(\varphi_n) = \int_0^\pi (\sin\theta_n)^n d\theta_n\, V(\varphi_{n-1})$ for $n \ge 3$. $V(\varphi_2) = 4\pi$.

(e) Note the fact: $I_n = \int_0^\pi (\sin\theta)^n d\theta = \frac{n-1}{n}\int_0^\pi (\sin\theta)^{n-2} d\theta = \frac{n-1}{n} I_{n-2}$ for $n \ge 2$.

$I_1 = 2$, $I_2 = \pi/2$, $I_0 = \pi$. $I_n = \frac{(n-1)!!}{n!!}\pi$ if n is even and $I_n = \frac{(n-1)!!}{n!!}2$ if n is odd.

So $V(\varphi_n) = I_n \cdots I_3 \cdot V(\varphi_2) = 4\prod_{k=1}^{n} I_k$. $(n \ge 2)$.

17.9 Denote $V_i = \frac{\partial\varphi}{\partial u_i}$ $(i=1,2)$ $N = V_1 \times V_2 / \|V_1 \times V_2\|$.

$A(\varphi) = \int_U \begin{vmatrix} V_1 \\ V_2 \\ V_1\times V_2 \end{vmatrix} \Big/ \|V_1\times V_2\| = \int_U (V_1\times V_2)\cdot(V_1\times V_2) / \|V_1\times V_2\| = \int_U \|V_1\times V_2\|$

17.10 (a) By Ex 14.9, W is normal vector field along φ. $\begin{vmatrix} E_1(p) \\ \vdots \\ E_n(p) \\ W \end{vmatrix} = \sum_{i=1}^{n} W_i^2 > 0$. So $W/\|W\|$ is the orientation vector field along φ

(b) $V(\varphi) = \int_U \begin{vmatrix} E_1(p) \\ \vdots \\ E_n(p) \\ W/\|W\| \end{vmatrix} = \int_U W\cdot W/\|W\| = \int_U \|W\|$

17.11 Let ~~$E = (e_1,\cdots,e_n)$, with~~ $e_i = (0,\cdots,0,1,0\cdots 0) \in R^n$, $A = (E_1^\psi, \cdots, E_n^\psi, N^\psi)$, $B = (E_1^{\varphi\circ h}, \cdots, E_n^{\varphi\circ h}, N^{\varphi\circ h})$

As N^φ is orientation vector field along φ. So $|B|>0$.

As $\psi = \varphi\circ h$, $A = (J^\varphi\circ h\cdot J_h\cdot e_1, \cdots, J^\varphi\circ h\cdot J_h\cdot e_n, N^\varphi\circ h) = (J^\varphi\circ h\cdot J_h, N^\varphi\circ h)$ if we assume $N^\psi = N^\varphi\circ h$.

Then $B = (J^\varphi\circ h, N^\varphi\circ h)$, $A^TB = \begin{pmatrix} J_h^T(J^\varphi\circ h)^T(J^\varphi\circ h) & 0 \\ 0 & 1\end{pmatrix}$ $B^TB = \begin{pmatrix} (J^\varphi\circ h)^T(J^\varphi\circ h) & 0 \\ 0 & 1\end{pmatrix}$

The zeros are because: $(N^\varphi\circ h)^T\cdot(J^\varphi\circ h)e_i = 0$ by definition of N^φ and $(N^\varphi\circ h)^T(J^\varphi\circ h)\cdot J_h e_i = 0$ by the fact that $\{e_1 \cdots e_n\}$ forms a basis of R^n so $J_h e_i$ can be written as a linear combination of $\{e_1\cdots e_n\}$.

So $|A^TB| = |J_h|\cdot|(J^\varphi\circ h)^T(J^\varphi\circ h)|$. $|B^TB| = |(J^\varphi\circ h)^T(J^\varphi\circ h)|$

So $|A|\cdot|B| = |J_h|\cdot|B|^2$. As $|J_h|>0$. $|B|>0$ so $|A| = |J_h||B|>0$

So $N^\psi = N^\varphi\circ h$ satisfies all the conditions to be orientation vector field.

17.12 (a) First prove $w(V_1\cdots V_i \cdots V_j \cdots V_K) = w(V_1\cdots, V_i, \cdots, V_j+\alpha V_i, \cdots V_K)$ where $\alpha\in R$.

This is because the latter $= w(V_1\cdots V_i\cdots V_j\cdots V_K) + \alpha\, W(V_1, \cdots V_i, \cdots, V_i, \cdots V_K)$ and $w(V_1\cdots V_i, \cdots V_i\cdots V_K) = 0$ by skewsymmetry.

If $\{V_1\cdots V_K\}$ is linearly dependent, then exist $\alpha_1\cdots\alpha_n \in R$, s.t $\sum_{i=1}^{K}\alpha_i V_i = 0$ and $\sum_{i=1}^{n}\alpha_i^2 \neq 0$. So ~~$W(V_1\cdots V_K) = w(V_1$~~ assume $\alpha_i \neq 0$, then

$w(V_1\cdots V_i\cdots V_K) = \frac{1}{\alpha_i} w(V_1 \cdots \alpha_i V_i, \cdots V_K) = \frac{1}{\alpha_i} w(V_1, \cdots, \alpha_i V_i + \alpha_1 V_1, \cdots V_K)$

$= \cdots = \frac{1}{\alpha_i} w(V_1, \cdots, \sum_{i=1}^{K}\alpha_i V_i, V_K) = 0$.

(b) If $K>n$, then $\{V_1\cdots V_K\}$ must be linearly dependent, so $w=0$.

17.13 (a) $\xi(V_1\cdots V_n)^2 = \left|\begin{pmatrix} V_1^T \\ \vdots \\ V_n^T \\ N\end{pmatrix}(V_1\cdots V_n N)\right| = \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1\end{vmatrix} = 1$ So $\xi(V_1\cdots V_n) = \pm 1$

and $\xi(V_1\cdots V_n) = 1$ iff $\{V_1 \cdots V_n\}$ is consistent with N.

(b) We only need to prove $w(U_1, \cdots, U_n) = w(V_1\cdots V_n)\cdot\xi(U_1\cdots U_n)$ for any $\{U_1\cdots U_n\}\in S_p$, and $V_1\cdots V_n$ is arbitrary orthonormal basis for S_p consistent with the orientation N on S. As $\{V_1\cdots V_n\}$ forms a basis of S_p, so there exist $\alpha_{ij}\in R$ s.t. $U_i = \sum_{j=1}^{n}\alpha_{ij}V_j$

So $w(U_1\cdots U_n) = \sum_{i_1\cdots i_n}\alpha_{1i_1}\cdots\alpha_{ni_n} w(V_{i_1}, \cdots, V_{i_n})$. If $i_p = i_q$ $(p\neq q)$ then $w(V_{i_1}\cdots V_{i_n}) = 0$

So $W(U_1\cdots U_n) = \sum_\sigma \alpha_{1\sigma(1)}\cdots\alpha_{n\sigma(n)} w(V_{\sigma(1)}, \cdots, V_{\sigma(n)})$ ~~$(\text{sign}\,\sigma)$~~. (1)

Likewise $\xi(U_1\cdots U_n) = \sum_\sigma \alpha_{1\sigma(1)}\cdots\alpha_{n\sigma(n)}\xi(V_{\sigma(1)}, \cdots, V_{\sigma(n)})$ ~~$(\text{sign}\,\sigma)$~~ (2)

by question (a)

Notice $W(V_{\sigma(1)}\cdots V_{\sigma(n)}) = (\text{sign}\,\sigma)\, W(V_1\cdots V_n)^{(3)}$, $\xi(V_{\sigma(1)}\cdots V_{\sigma(n)}) = (\text{sign}\,\sigma)\,\xi(V_1\cdots V_n) = \text{sign}\,\sigma^{(4)}$

So continuing (1) and plugging (3)(4) into (1) we have $W(U_1\cdots U_n) = \sum_\sigma \alpha_{1\sigma(1)}\cdots\alpha_{n\sigma(n)}\left(\xi(V_{\sigma(1)}, \cdots, V_{\sigma(n)})\, W(V_1\cdots V_n)\right) = W(V_1\cdots V_n)\cdot\sum_\sigma\alpha_{1\sigma(1)}\cdots\alpha_{n\sigma(n)}\xi(V_{\sigma(1)}\cdots V_{\sigma(n)})$

$\overset{\text{by}(2)}{=} W(V_1\cdots V_n)\,\xi(U_1, \cdots U_n)$

17.14 (a) ~~linear~~ multilinearity is obvious. We only need to prove skewsymmetry. To this end, we only need to prove for $\forall\, i,j \in \{1,\cdots,k+l\}$, $(w_1\wedge w_2)(v_1\cdots v_i\cdots v_j\cdots v_{k+l})$ $= -(w_1\wedge w_2)(v_1\cdots v_j\cdots v_i\cdots v_{k+l})$.

For $\forall \sigma$, ~~if~~ let p,q s.t. $\sigma(p)=i, \sigma(q)=j$. If $p,q\le k$, then v_i, v_j both appear in w_1 under such σ, so swapping $v_i\ v_j$ will just inverse the sign. The same happens if $p,q>k$.

If $p\le k$, $q>k$, then look at $\hat\sigma$ which is the same as σ except $\hat\sigma(p)=j$, $\hat\sigma(q)=i$.

So $\operatorname{sign}\hat\sigma = -\operatorname{sign}\sigma$. For $(w_1\wedge w_2)(\underset{\operatorname{sign}\hat\sigma}{v_1}\cdots v_i\cdots v_j\cdots v_{k+l})$ we have summands

$(\operatorname{sign}\sigma)\, w_1(\cdots v_i\cdots)\, w_2(\cdots v_j\cdots) - (\operatorname{sign}\sigma)\, w_1(\cdots v_j\cdots)\, w_2(\cdots v_i\cdots)$.

For $(w_1\wedge w_2)(v_1\cdots v_j\cdots v_i\cdots v_{k+l})$, we have summands

$(\operatorname{sign}\sigma)\, w_1(\cdots v_j\cdots)\, w_2(\cdots v_i\cdots) - (\operatorname{sign}\sigma)\, w_1(\cdots v_i\cdots)\, w_2(\cdots v_j\cdots)$.

So ~~for~~ the summands for swapped $v_i\, v_j$ ~~are~~ have opposite sign.

This also happens to $p>k$, $q\le k$. So in all $(w_1\wedge w_2)(v_1\cdots v_i\cdots v_j\cdots v_{k+l}) = -(w_1\wedge w_2)(v_1\cdots v_j\cdots v_i\cdots v_n)$.

(b) We only need to prove that if

$\big(\sigma(1)\cdots\sigma(k),\sigma(k+1),\cdots\sigma(k+l)\big) = \big(\hat\sigma(l+1)\cdots\hat\sigma(k+l),\hat\sigma(1),\cdots,\sigma(l)\big)$, i.e.

$w_1(v_{\sigma(1)}\cdots v_{\sigma(k)})\cdot w_2(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)}) = w_2(v_{\sigma(1)}\cdots v_{\sigma(l)})\cdot w_1(v_{\sigma(l+1)}\cdots v_{\sigma(l+k)})$, then

$\operatorname{sign}\sigma = (-1)^{kl}\operatorname{sign}\hat\sigma$. This boils down to how many number of swaps is needed in order to change $(a_1\cdots a_k a_{k+1}\cdots a_{k+l})$ to $(a_{k+1}\cdots a_{k+l}\, a_1\cdots a_k)$, and we only care about the odd/even of the number. One schedule is pushing a_{k+1} ahead ~~for~~ by swapping with the element to its left for k times, i.e. $(a_1\cdots a_{k-1}a_k a_{k+1}) \to (a_1\cdots a_{k-1}a_{k+1}a_k) \to (a_1\cdots a_{k+1}a_{k-1}a_k) \to \cdots \to (a_{k+1}\, a_1\cdots a_k\cdots)$. Doing the same for $a_{k+2},\cdots a_{k+l}$, then we change ~~$(a_1\cdots a_{k+l})$~~ $(a_1\cdots a_k a_{k+1}\cdots a_{k+l})$ to $(a_{k+1}\cdots a_{k+l}\, a_1\cdots a_k)$ in kl steps.

Since the odd/even of step number is independent of schedule.

we proved $\operatorname{sign}\sigma = (-1)^{kl}\operatorname{sign}\hat\sigma$.

(c) $w_1\wedge(w_2+w_3) = \frac{1}{k!\,l!}\sum_\sigma(\operatorname{sign}\sigma)\, w_1(v_{\sigma(1)}\cdots v_{\sigma(k)})(w_2+w_3)(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)})$

$= \frac{1}{k!\,l!}\sum_\sigma(\operatorname{sign}\sigma)\, w_1(v_{\sigma(1)}\cdots v_{\sigma(k)})\, w_2(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)}) + \frac{1}{k!\,l!}\sum_\sigma(\operatorname{sign}\sigma)\, w_1(v_{\sigma(1)}\cdots v_{\sigma(k)})\, w_3(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)})$

$= (w_1\wedge w_2) + (w_1\wedge w_3)$.

(d) $(w_1\wedge w_2)\wedge w_3 = \frac{1}{k!\,l!\,m!\,(k+l)!}\sum_{\sigma,\hat\sigma}(\operatorname{sign}\sigma)(\operatorname{sign}\hat\sigma)\, w_1(v_{\sigma(\hat\sigma(1))},\cdots v_{\sigma(\hat\sigma(k))})\, w_2(v_{\sigma(\hat\sigma(k+1))}\cdots v_{\sigma(\hat\sigma(k+l))})\, w_3(v_{\sigma(k+l+1)}\cdots v_{\sigma(k+l+m)})$ (*)

where σ is a permutation of $1\cdots(k+l+m)$ and $\hat\sigma$ is a permutation of $1\cdots k+l$.

Notice $(\operatorname{sign}\sigma)\cdot(\operatorname{sign}\hat\sigma) = \operatorname{sign}(\sigma\circ\hat\sigma)$. (we can define $\hat\sigma(i)=i$ for $i>k+l$).

For each $w_1(v_{i_1}\cdots v_{i_k})\, w_2(v_{i_{k+1}}\cdots v_{i_{k+l}})\, w_3(v_{i_{k+l+1}}\cdots v_{i_{k+l+m}})$, there exist $(k+l)!$ different combinations of σ and $\hat\sigma$ which finally results in this order of subscript by permutating from $(1,\cdots,k+l+m)$. In fact, for any $\hat\sigma$, there exists a unique σ, such that $\sigma\circ\hat\sigma$ yields above subscripts. Besides, all such combinations have the same sign of $\sigma\circ\hat\sigma$. So (*) is equal to $\frac{1}{k!\,l!\,m!}\sum_{\sigma} w_1(v_{\sigma(1)}\cdots v_{\sigma(k)})\, w_2(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)})\, w_3(v_{\sigma(k+l+1)}\cdots v_{\sigma(k+l+m)})$ (1)

For the same reason, $w_1\wedge(w_2\wedge w_3)$ is also equal to (1).

Thus $(w_1\wedge w_2)\wedge w_3 = w_1\wedge(w_2\wedge w_3)$.

(e) First prove for $\forall k\in[1,n]$ $(w_1\wedge\cdots\wedge w_k)(X_1\cdots X_k)=1$. by induction

If $k=1$, then $w_1(X_1)=X_1(p)\cdot X_1(p)=1$. If it's true for k, then

$(w_1\wedge\cdots\wedge w_{k+1})(X_1\cdots X_{k+1}) = \frac{1}{k!}\sum_{\sigma}(\text{sign}\,\sigma)(w_1\wedge\cdots\wedge w_k)(X_{\sigma(1)}\cdots X_{\sigma(k)})\, w_{k+1}(X_{\sigma(k+1)})$

If $\sigma(k+1)\neq k+1$, then $w_{k+1}(X_{\sigma(k+1)}) = X_{k+1}\cdot X_{\sigma(k+1)}=0$. So we

only look at those σ, s.t. $\sigma(k+1)=k+1$. so $\sigma(1)\cdots\sigma(k)$ is a permutation of $1,2,\cdots,k$ ~~Let $\tilde\sigma(i)=\sigma(i)$ $i=1\cdots k$~~. Then

$(w_1\wedge\cdots\wedge w_{k+1})(X_1\cdots X_{k+1}) = \frac{1}{k!}\sum_{\sigma}(\text{sign}\,\sigma)^2\,(w_1\wedge\cdots\wedge w_k)(X_1\cdots X_k)$. as $w_1\wedge\cdots\wedge w_k$ is k-form

$=\frac{1}{k!}\cdot k! = 1$ (Implicitly using the fact that when a k+1 permutation σ satisfies $\sigma(k+1)=k+1$, then its sign is equal to the k permutation $\tilde\sigma$ defined as $\tilde\sigma(i)=\sigma(i)$ $i=1,\cdots,k$.)

So $(w_1\wedge\cdots\wedge w_k)(X_1\cdots X_k)=1$ for all $k=1\cdots n$.

This step is not necessary. We just follow the hint on textbook.

Next prove for $\forall k\in[1,n]$, $i>k$, $X_i\lrcorner(w_1\wedge\cdots\wedge w_k)=0$. Actually $\forall v_1\cdots v_{k-1}\in S_p$,
$X_i\lrcorner(w_1\cdots w_k)(v_1\cdots v_{k-1}) = (w_1\wedge\cdots\wedge w_k)(X_i(p), v_1,\cdots,v_{k-1})$. Expanding as in the definition,
if $X_i(p)$ appear in $w_k(\cdot)$, then $w_k(X_i(p)) = X_k(p)\cdot X_i(p)=0$
If $X_i(p)$ appear in $w_1\wedge\cdots\wedge w_{k-1}$, then by some induction like proof, it's easy
to show $(w_1\wedge\cdots\wedge w_k)(\cdots,X_i(p),\cdots)=0$. So $X_i\lrcorner(w_1\cdots w_k)=0$, $i>k$, $k\in[1,n]$

Finally, as $\{X_1\cdots X_{n+1}\}$ is an orthonormal basis for S_p, by Ex 17.13, $f(p)=1$ because $(w_1\wedge\cdots\wedge w_n)(X_1(p)\cdots X_n(p))=1$. So $w_1\wedge\cdots\wedge w_n = \xi$

17.15 (a) multilinearity is obvious. $f^*w(v_{\sigma(1)},\cdots,v_{\sigma(k)}) = w(df(v_{\sigma(1)}),\cdots,df(v_{\sigma(k)}))$

$=(\text{sign}\,\sigma)\,w(df(v_1),\cdots,df(v_k)) = (\text{sign}\,\sigma) f^*w(v_1,\cdots,v_k)$

As w, df are smooth, f^*w is also smooth.

(b) $\int_{\varphi} f^*w = \int_U w(df(E_1^{\varphi}),\cdots,df(E_k^{\varphi})) = \int_U w(E_1^{f\circ\varphi},\cdots E_k^{f\circ\varphi}) = \int_{f\circ\varphi} w$

(c) Suppose $\{f_i\}$ is a partition of unity on S subordinate to a collection $\{\varphi_i\}$ of one to one local parametrizations of S. We prove first that $\{f_i\circ f^{-1}\}$ is a partition of unity of $\tilde S$, subordinate to a collection $\{f\circ\varphi_i\}$ of one to one local parametrizations of $\tilde S$.

① $\forall q\in\tilde S$, $f^{-1}(q)\in S$ (as f is diffeomorphism), So $f_i(f^{-1}(q))\geq 0$ $i=1\cdots m$

② $\forall q\in\tilde S$, $f^{-1}(q)\in S$, thus $\sum_{i=1}^{m} f_i(f^{-1}(q))=1$

③ as f,φ_i are both one to one, so $f\circ\varphi_i$ is one to one. As φ_i is regular and m=n and f ~~is dif~~ is diffeomorphism, so Jf is invertible and $f\circ\varphi_i$ is regular. So $f\circ\varphi_i$ is also local parametrization of $\tilde S$. Besides, f is orientation preserving and $f\circ\varphi_i$ must be open as f^{-1} is smooth ~~and invertible~~

Suppose f_i is identically zero outside the image under φ_i of a compact subset

B_i of U_i. ~~then as f is smooth, $f(B_i)$ is also compact. $\forall x \notin f(B_i)$,~~ and $x \in$

let $x \notin f(\varphi_i(B_i))$. $(x \in f(\varphi_i(U_i)))$, then if $f_i(f^{-1}(x)) \neq 0$, then $f^{-1}(x) \in \varphi_i(B_i)$, then $x = f(f^{-1}(x)) \in f(\varphi_i(B_i))$, which contradicts with our assumption. so $f_i(f^{-1}(x)) = 0$

So $f_i \circ f^{-1}$ is identically 0 outside the image under $f \circ \varphi_i$ of a compact subset B_i of U_i.

Combining ①-③, we conclude $\{f_i \circ f^{-1}\}$ is a partition of unity of $\tilde{S}$, subordinate to a collection $\{f \circ \varphi_i\}$ of one-to-one local parametrization of $\tilde{S}$.

Finally. $\int_S f^* w = \sum_i \int_{\varphi_i} f_i f^* w = \sum_i \int_{U_i} f_i \circ \varphi_i \cdot w(df(E_1^{\varphi_i}), \dots, df(E_k^{\varphi_i}))$

$$= \sum_i \int_{U_i} f_i \circ f^{-1} \circ f \circ \varphi_i \cdot w(E_1^{f \circ \varphi_i}, \dots, E_k^{f \circ \varphi_i}) = \int_{f \circ \varphi_i} f_i \circ f^{-1} w = \int_{\tilde{S}} w$$

~~17.16~~

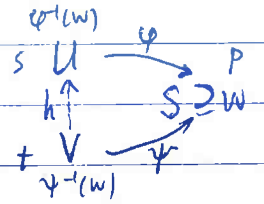
17.16 For $\forall p \in S^n$. If $v_1 \cdots v_n \in S_p$ is a basis of S_p and $\begin{vmatrix} v_1 \\ \vdots \\ v_n \\ N(p) \end{vmatrix} > 0$. then $df(v_i) = -v_i$, $N(f(p)) = -N(p)$, so $\begin{vmatrix} df(v_1) \\ \vdots \\ df(v_n) \\ N(f(p)) \end{vmatrix} = (-1)^{n+1} \begin{vmatrix} v_1 \\ \vdots \\ v_n \\ N(p) \end{vmatrix}$, which is positive iff n is odd.

17.17 (a) $\lim_{t \to 0^+} h(t) = 0$, $h'(t) = \frac{1}{t^2} e^{-\frac{1}{t}}$, so $\lim_{t \to 0^+} h'(t) = 0$. Generally, $h^{(n)}(t)$ must be in the form of $h^{(n)}(t) = P(\frac{1}{t}) \cdot e^{-\frac{1}{t}}$, where $P(x)$ is a polynomial function of x, with finite degree. ~~Actually~~. So $\lim_{t \to 0^+} h^{(n)}(t) = 0$. Obviously $\lim_{t \to 0^-} h^{(n)}(t) = 0$ So h is smooth

(b) $h_r(t) = h(u(t))$, where $u(t) = r^2 - t^2$. Since both $u(t)$, $h(u)$ are smooth, so $h_r(t)$ is smooth.

In the proof of Thm 4. φ_p^{-1} is smooth, $u(q) = -\|q - \varphi_p^{-1}(p)\|^2 + r_p^2$ is also smooth wrt $q \in \mathbb{R}^{n+1}$

So $g_p(q) = h(u(\varphi_p^{-1}(q)))$ is smooth.



17.18 Since φ, ψ are both one to one local parametrization

So $\varphi|_{\varphi^{-1}(W)}$ and $\psi|_{\psi^{-1}(W)}$ are both bijective ~~to W~~ from $\varphi^{-1}(W)$ or $\psi^{-1}(W)$ to W. So φ^{-1} and ψ^{-1} are bijective, thus $\varphi^{-1} \circ \psi|_{\psi^{-1}(W)}$ is also bijective. φ and ψ are both smooth and regular, so φ^{-1}, ψ^{-1} must be smooth. So $\varphi^{-1} \circ \psi|_{\psi^{-1}(W)}$ is smooth, and its inverse $\psi^{-1} \circ \varphi|_{\varphi^{-1}(W)}$ is also smooth. So $\varphi^{-1} \circ \psi|_{\psi^{-1}(W)}$ is diffeomorphism.

The textbook only defines "orientation preserving" for a map between two oriented n-surfaces in $\mathbb{R}^{n+1}$ at a regular point, so we don't know what it means by h being orientation preserving, because h maps from $\psi^{-1}(W)$ (open set) to $\varphi^{-1}(W)$ (open set). However we can still prove that $|J_h| > 0$, thus $\psi|_{\psi^{-1}(W)}$ is reparametrization of $\varphi|_{\varphi^{-1}(W)}$.

For any point $p \in W$, suppose $s = \varphi^{-1}(p)$, $t = \psi^{-1}(p)$. Since both φ and ψ are local parametrizations of S, we have $s = \varphi^{-1}(\psi(t)) = h(t)$. and

$A = \begin{pmatrix} (J_\varphi(s)\cdot e_1)^T \\ \vdots \\ (J_\varphi(s) e_n)^T \\ N^T(p) \end{pmatrix}$, $|A|>0$, $B = \begin{pmatrix} (J_\psi(t) e_1)^T \\ \vdots \\ (J_\psi(t) e_n)^T \\ N^T(p) \end{pmatrix}$, $|B|>0$. But $J_\psi(t) = J_{\varphi\circ h}(t)\cdot J_h(t) = J_\varphi(s) J_h(t)$ as $\psi = \varphi\circ h$.

where $N(p)$ is the orientation of S.

$AB^T = \begin{pmatrix} J_\varphi^T(s) \\ N^T(p) \end{pmatrix} (J_\varphi(s) J_h(t), N(p)) = \begin{pmatrix} J_\varphi^T(s)\cdot J_\varphi(s)\cdot J_h(t) & 0 \\ 0 & 1 \end{pmatrix}$ So $|A|\cdot|B| = |J_\varphi^T(s) J_\varphi(s)|\cdot|J_h(t)|$

As $J_\varphi^T J_\varphi$ is positive semi-definite, $|J_\varphi^T(s) J_\varphi(s)| \geq 0$. But $|A|,|B|>0$. So $|J_h(t)|>0$.

Since p is any point on W and ψ is bijective, so $|J_h(t)|>0$ for any $t\in\psi^{-1}(W)$

Thus $\psi|_{\psi^{-1}(W)} = \varphi\circ h|_{\psi^{-1}(W)}$ is reparametrization of $\varphi|_{\varphi^{-1}(W)}$

17.19 Denote $x = X(p) = (x_1, x_2, x_3)$, $y = Y(p) = (y_1, y_2, y_3)$

$(W_X\wedge W_Y)(v,w) = W_X(v)W_Y(w) - W_X(w)W_Y(v) = (x\cdot v)\cdot(y\cdot w) - (x\cdot w)(y\cdot v)$

$= \left(\sum_{i=1}^3 x_i v_i\right)\left(\sum_{i=1}^3 y_i w_i\right) - \left(\sum_{i=1}^3 x_i w_i\right)\left(\sum_{i=1}^3 y_i v_i\right)$

$(X\times Y)(p)\cdot(v\times w) = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)\cdot(v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1)$

$= x_2v_2y_3w_3 + x_3v_3y_2w_2 + x_3v_3y_1w_1 + x_1v_1y_3w_3 + x_1v_1y_2w_2 + x_2v_2y_1w_1$

$- x_2y_3v_3w_2 - x_3y_2v_2w_3 - x_3y_1v_1w_3 - x_1y_3v_3w_1 - x_1y_2v_2w_1 - x_2y_1v_1w_2$

$= \left(\sum_{i=1}^3 x_i v_i\right)\left(\sum_{i=1}^3 y_i w_i\right) - \left(\sum_{i=1}^3 x_i w_i\right)\left(\sum_{i=1}^3 y_i v_i\right)$

So $(W_X\wedge W_Y)(v,w) = (X\times Y)(p)\cdot(v\times w)$

18.1 $\varphi(t,\theta) = (t, y(t)\cos\theta, y(t)\sin\theta)$. $t\in I$, $\theta\in[0,2\pi)$

$E_1^\varphi = \frac{\partial\varphi}{\partial t} = (1, y'\cos\theta, y'\sin\theta)$, $E_2^\varphi = \frac{\partial\varphi}{\partial\theta} = (0, -y\sin\theta, y\cos\theta)$

$N = E_1^\varphi\times E_2^\varphi / \|E_1^\varphi\times E_2^\varphi\| = (1+y'^2)^{-1/2}(y', -\cos\theta, -\sin\theta)$

$L_p(E_1(p)) = -\frac{\partial N}{\partial t} = \left(\frac{-y''}{(1+y'^2)^{3/2}}, \frac{-y''y'\cos\theta}{(1+y'^2)^{3/2}}, \frac{-y''y'\sin\theta}{(1+y'^2)^{3/2}}\right) = -y''(1+y'^2)^{-3/2}(1, y'\cos\theta, y'\sin\theta)$

$L_p(E_2(p)) = -\frac{\partial N}{\partial\theta} = \left(0, \frac{-\sin\theta}{(1+y'^2)^{1/2}}, \frac{\cos\theta}{(1+y'^2)^{1/2}}\right) = -\frac{1}{y(1+y'^2)^{1/2}}(0, -y\sin\theta, y\cos\theta)$

So $k_1(t,\theta) = -y''(t)/(1+y'^2)^{3/2}$, $k_2(t,\theta) = -\left(y(1+y'^2)^{\frac{1}{2}}\right)^{-1}$

18.2 $E_1 = \frac{\partial\varphi}{\partial t} = (\cos\theta, \sin\theta, 0)$, $E_2 = \frac{\partial\varphi}{\partial\theta} = (-t\sin\theta, t\cos\theta, 1)$

$N = \frac{1}{\sqrt{1+t^2}}(\sin\theta, -\cos\theta, t)$. $L_p(E_1) = -\frac{\partial N}{\partial t} = \frac{-1}{(1+t^2)^{3/2}}(-t\sin\theta, t\cos\theta, 1) = -(1+t^2)^{3/2}E_2$

$L_p(E_2) = -\frac{\partial N}{\partial\theta} = -(1+t^2)^{-1/2}(\cos\theta, \sin\theta, 0) = -(1+t^2)^{-1/2}E_1$

So the matrix of L_p wrt E_1, E_2 is $\begin{pmatrix} 0 & -(1+t^2)^{-3/2} \\ -(1+t^2)^{-1/2} & 0 \end{pmatrix}$. $H=0$.

18.3 By Ex 10.1. Let $\alpha(t) = (x(t), y(t))$, $t\in I$, then $k\circ\alpha = (x'y'' - x''y')/(x'^2+y'^2)^{3/2}$ ~~$\neq 0$~~

If $k\equiv 0$, then $x'y'' - y'x'' = 0$. Since α is regular, so either $x'\neq 0$ or $y'\neq 0$

Suppose $y'\neq 0$, ~~then~~ in some subinterval of I, then $\left(\frac{x'}{y'}\right)' = 0$, $\frac{x'}{y'} = C_1$, $x - C_2 = C_1(y - C_3)$

So $X = C_1 Y + C_2$. Suppose $X' \neq 0$ in some subinterval of I, similarly $Y = C_1' X + C_2'$.
It is obvious that a line segment parallel to x_1-axis and a line segment parallel to x_2-axis do not fit together smoothly, so S is a segment of a straight line.

18.4. Suppose the two principal curvatures are k_1, k_2. Then minimal surface $\Rightarrow k_1 + k_2 = 0$
So $K = k_1 * k_2 \leq 0$.

18.5 Suppose the Weingarten map L_p has two eigenvalues λ_1, λ_2 corresponding to two eigenvectors V_1, V_2 which are orthonormal. $\forall \hat{V} \in S_p$, $\exists \alpha_1, \alpha_2 \in \mathbb{R}$, s.t. $\hat{V} = \alpha_1 V_1 + \alpha_2 V_2$.
So $k(\hat{V}) = L_p(\hat{V}) \cdot \hat{V} = (\alpha_1 \lambda_1 V_1 + \alpha_2 \lambda_2 V_2)(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1^2 \lambda_1 + \alpha_2^2 \lambda_2$
As S is minimal surface, so $\lambda_1 + \lambda_2 = 0$, $\lambda_2 = -\lambda_1$, $k(\hat{V}) = \lambda_1 (\alpha_1^2 - \alpha_2^2)$
Now Let $V = \frac{\sqrt{2}}{2}(V_1 + V_2)$, $W = \frac{\sqrt{2}}{2}(V_1 - V_2)$, then $V W = 0$.
$k(V) = \lambda_1 \left(\left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 \right) = 0$. $k(W) = \lambda_1 \left(\left(\frac{\sqrt{2}}{2}\right)^2 - \left(-\frac{\sqrt{2}}{2}\right)^2 \right) = 0$.

18.6 $\underbrace{\quad}_{\forall v \in S_p}$ $dN_p(v) = \nabla_v N\big|_p = -L_p(v)$. Suppose the principal curvatures are λ_1, λ_2 corresponding to principal curvature directions V_1, V_2. Since V_1, V_2 span S_p, so $\exists \alpha_1, \alpha_2 \in \mathbb{R}$, $V = \alpha_1 V_1 + \alpha_2 V_2$
$\|dN_p(v)\| = \|L_p(v)\| = \|L_p(\alpha_1 V_1 + \alpha_2 V_2)\| = \|\alpha_1 \lambda_1 V_1 + \alpha_2 \lambda_2 V_2\| = \sqrt{\alpha_1^2 \lambda_1^2 + \alpha_2^2 \lambda_2^2}$
As S is minimal surface, $\lambda_1 = -\lambda_2$, So $\|dN_p(v)\| = |\lambda_1| \sqrt{\alpha_1^2 + \alpha_2^2} = |\lambda_1| \cdot \|v\|$.

18.7 $\cancel{\frac{d}{ds}\Big|_0 \ell(\alpha_s) = \frac{d}{ds}\Big|_0 \int_a^b \|\dot{\alpha}(s)\| dt = \int_a^b \frac{d}{ds} \|\dot{\alpha}(s)\|}$
$\frac{d}{ds}\Big|_0 \ell(\alpha_s) = \frac{d}{ds}\Big|_0 \int_a^b \|\dot{\alpha}_s(t)\| dt = \int_a^b \frac{d}{ds}\Big|_0 \|\dot{\alpha}_s(t)\| dt$ $\quad (*)$
$\frac{d}{ds}\Big|_0 \|\dot{\alpha}_s(t)\| = \frac{d}{ds}\Big|_0 \sqrt{\dot{\alpha}_s(t) \cdot \dot{\alpha}_s(t)} = 2 \dot{\alpha}_s(t) \frac{d}{ds} \dot{\alpha}_s(t) / 2\|\dot{\alpha}_s(t)\| \Big|_{s=0}$
As $\|\dot{\alpha}_s(t)|_{s=0}\| = \|\dot{\alpha}(t)\| = 1$, $\dot{\alpha}_s(t)\big|_{s=0} = \dot{\alpha}(t)$
$\frac{d}{ds}\Big|_0 \dot{\alpha}_s(t) = \frac{\partial^2 \psi(t,0)}{\partial s \partial t}$. So $\frac{d}{ds}\Big|_0 \|\dot{\alpha}(t)\| = \dot{\alpha}(t) \cdot \frac{\partial^2 \psi(t,0)}{\partial s \partial t}$. Plugging into $(*)$
$\frac{d}{ds}\Big|_0 \ell(\alpha_s) = \int_a^b \dot{\alpha}(t) \cdot \frac{\partial^2 \psi(t,0)}{\partial s \partial t} dt = \int_a^b \dot{\alpha}(t)\, d \frac{\partial \psi(t,0)}{\partial s} = \int_a^b \dot{\alpha}(t)\, d X(t)$
$= \dot{\alpha}(t) X(t) \Big|_a^b - \int_a^b \ddot{\alpha}(t) X(t) dt$ (Note $X(t) = \frac{\partial \psi(t,s)}{\partial s}\Big|_{s=0} := \frac{\partial \psi(t,0)}{\partial s}$)
Using Ex 10.6, $\ddot{\alpha}(t) = k(t) N$, we have $\frac{d}{ds}\Big|_0 \ell(\alpha_s) = \dot{\alpha}(t) X(t) \Big|_a^b - \int_a^b (X \cdot N) k(t) dt$
If $\psi(a,s) = \alpha(a)$, $\psi(b,s) = \alpha(b)$ $\forall s$ (i.e., compactly supported), then
$X(a) = 0 = X(b)$ So $\frac{d}{ds}\Big|_0 \ell(\alpha_s) = -\int_a^b (X \cdot N) k(t) dt$.

22.1 Example 1: $\|\psi(p)-\psi(q)\| = \|p+a-(q+a)\| = \|p-q\|$

Example 2: $\|\psi(p)-\psi(q)\| = \|Ap-Aq\| = \|A(p-q)\| = \|p-q\|$, as $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. $\|A\begin{pmatrix}x_1\\x_2\end{pmatrix}\| =$
$\|(\cos\theta x_1 - \sin\theta x_2, \sin\theta x_1 + \cos\theta x_2)\| = [(\cos\theta x_1 - \sin\theta x_2)^2 + (\sin\theta x_1 + \cos\theta x_2)]^{1/2} = (x_1^2+x_2^2)^{1/2}$

Example 3: $\|\psi(p)-\psi(q)\| = \| p+2(b-p\cdot a)\cdot a - q - 2(b-q\cdot a)a\| = \|p-q-2[(p-q)\cdot a]\cdot a\|$, let $x = p-q$
$= [(x-2(x\cdot a)\cdot a)^T(x-2(x\cdot a)\cdot a)]^{1/2} = [x^Tx + 4(x\cdot a)^2 - 4(x\cdot a)^2]^{1/2} = \|x\| = \|p-q\|$.

22.2. $\forall x \in R^{n+1}$, $\psi_1(\psi_2(x)) = \psi_1(x+a) \overset{\psi_1 \text{ is linear}}{=} \psi_1(x) + \psi_1(a) = \tilde{\psi}_2(\psi_1(x))$, $\tilde{\psi}_2(\hat{x}) = \hat{x} + \psi_1(a)$.

22.3(a) $\psi(v)\cdot\psi(w) = v\cdot w \Rightarrow \psi(v)\cdot\psi(v) = v\cdot v \Rightarrow \|\psi(v)\| = \|v\|$.
$\|\psi(v)\| = \|v\| \Rightarrow \psi(v)\cdot\psi(w) = \frac{1}{2}[\|\psi(v+w)\|^2 - \|\psi(v)\|^2 - \|\psi(w)\|^2] =$
$= \frac{1}{2}[\|v+w\|^2 - \|v\|^2 - \|w\|^2] = v\cdot w$

(b) ① $\forall$ orthonormal basis $\{e_1 \cdots e_n\}$. let $v = \sum_{i=1}^n v_i e_i$, then if $\{\psi(e_1) \cdots \psi(e_{n+1})\}$ is orthonormal
we have $\|\psi(v)\| = \|\psi(\sum_{i=1}^n v_i e_i)\| = \|\sum_{i=1}^n v_i \psi(e_i)\| = \sqrt{\sum_{i=1}^n v_i^2} = \|v\|$

② By (a), If $\{e_1 \cdots e_n\}$ is orthonormal, then $\psi(e_i)\cdot\psi(e_j) = e_i\cdot e_j = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$
so $\{\psi(e_1), \cdots, \psi(e_n)\}$ is orthonormal basis for R^{n+1}

(c) ④ Let $\psi(e_i) = \sum_{j=1}^n a_{ij} e_j$, where $\{e_1 \cdots e_n\}$ is an orthonormal basis of R^{n+1}.
① If A is orthogonal, then letting $P = (\psi(e_1), \cdots, \psi(e_n)) = AQ$ where $Q = (e_1 \cdots e_n)$,
we have $P^T\cdot P = Q^TA^TAQ = Q^TQ = I$, so $\psi(e_1) \cdots \psi(e_n)$ is orthonormal
By (b) we have ψ is orthogonal transformation.
② If ψ is orthogonal, then by (b) $P = (\psi(e_1) \cdots \psi(e_n))$ is also orthonormal
$I = P'P = AQQ^TA^T = AA^T$ so A is orthogonal.

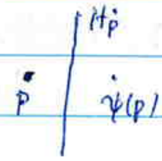
22.4 (a) By Ex 22.3(c). The matrix is orthonormal $\Leftrightarrow$ orthogonal linear transformation
So rotation $\Leftrightarrow \begin{vmatrix}x_1 & x_3\\ x_2 & x_4\end{vmatrix} = 1$ and $A^TA = I$ where $A = \begin{pmatrix}x_1 & x_2\\ x_2 & x_4\end{pmatrix}$.
$\Leftrightarrow x_1^2 + x_3^2 = 1$, $x_1x_2 + x_3x_4 = 0$, $x_2^2 + x_4^2 = 1$, $x_1x_4 - x_2x_3 = 1$ $(*)$
Let $x_1 = \cos\theta$, $x_3 = -\sin\theta$, $x_2 = \cos\varphi$, $x_4 = \sin\varphi$, we have
$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi = x_1x_2 + x_3x_4 = 0$
$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi = -x_3x_2 + x_1x_4 = +1$
So $\theta+\varphi = 2k\pi + \frac{\pi}{2}$, $\sin\varphi = \cos\theta$, $\cos\varphi = \sin\theta$, so $A = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$
obviously, A in such a form must satisfy $(*)$

(b) $\forall$ eigenvalue and eigenvector α_i, λ_i: $\psi\alpha_i = \lambda_i\alpha_i$, then $\alpha_i^T\psi^T\psi\alpha_i = \alpha_i^T\lambda_i^2\alpha_i$
As $\psi^T\psi = I$ by Ex 22.3(c), $1 = \alpha_i^T\alpha_i = \lambda_i^2$. So $\lambda_i = \pm 1$. If all λ_i are -1
then $|\psi| = \prod_{i=1}^3 \lambda_i = -1$, violating definition of rotation. So $\exists_{\lambda_i = 1} \alpha_i$: $\psi\alpha_i = \alpha_i$.

(c) For $\forall v\perp e_1$, $\psi(v)\cdot\psi(e_1)=v\cdot e_1=0$, So $v\perp e_1$, So the matrix must be in the form of $A=\begin{pmatrix}1&0&0\\0&x_1&x_3\\0&x_2&x_4\end{pmatrix}$. A orthonormal $\Leftrightarrow$ $\begin{pmatrix}x_1&x_3\\x_2&x_4\end{pmatrix}$ orthonormal, $\begin{vmatrix}x_1&x_3\\x_2&x_4\end{vmatrix}=|A|=1$.
So by the proof in (a), $A=\begin{pmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{pmatrix}$.

22.5 Map: $\forall(x_1,x_2)$ on $x_1x_2=1$ to $\varphi(x_1,x_2)=\left(\frac{\sqrt2}{2}(x_1+x_2),\frac{\sqrt2}{2}(x_1-x_2)\right)$.
Obviously, $\varphi(x_1,x_2)$ is on $x_1^2-x_2^2=2$. $\|\varphi(x_1,x_2)-\varphi(x_1',x_2')\|=$
$\left\|\left(\frac{\sqrt2}{2}(x_1+x_2-x_1'-x_2'),\frac{\sqrt2}{2}(x_1-x_2-x_1'+x_2')\right)\right\|=\sqrt{\frac12(m+n)^2+\frac12(m-n)^2}$ $(m\triangleq x_1-x_1',\ n\triangleq x_2-x_2')$
$=\sqrt{m^2+n^2}=\|(x_1-x_1',x_2-x_2')\|$ So φ is rigid motion.

22.6 (a) $0=\|x-\psi(p)\|^2-\|x-p\|^2=2(p-\psi(p))\cdot x+\|\psi(p)\|^2-\|p\|^2$ (∗)



As $p\bar\in F$, so $p-\psi(p)\neq 0$. So H_p is hyperplane

(b) $\forall q\in F$, $\|q-\psi(p)\|=\|\psi(q)-\psi(p)\|=\|q-p\|$. So $q\in H_p$, so $F\subseteq H_p$.

(c) ~~By (∗) in (a), $p-\psi(p)\perp H_p$~~, Obviously, $q=\frac12(\psi(p)+p)\in H_p$
By (∗) $q-p=\frac12(\psi(p)-p)\perp H_p$, $q-\psi(p)=\frac12(p-\psi(p))\perp H_p$.
So the line segment $p\to\psi(p)$ intersects with H_p perpendicularly at q.
As $\|q-p\|=\|q-\psi(p)\|$, $\psi_p(\psi(p))=p$ i.e. p is fixed point of $\psi_p\circ\psi$.
Besides, as $F\subset H_p$ and $\psi(F)\subseteq F$ and ψ_p is reflection through H_p, it is obvious that $F\subseteq\{$fixed point of $\psi_p\circ\psi\}$.

(d) Suppose $\psi=\psi_2\circ\psi_1$ is the unique decomposition of ψ into an orthogonal transformation ψ_1 followed by translation ψ_2. As $\psi(0)=0$, ψ_2 is identity. So
$\psi\left(\sum_{i=1}^k c_ip_i\right)\underset{\text{as } \psi_2=\text{identity}}{=}\psi_1\left(\sum_{i=1}^k c_ip_i\right)\underset{\text{linearity of } \psi_1}{=}\sum_{i=1}^k c_i\psi_1(p_i)\underset{\psi_2 \text{ is identity}}{=}\sum_{i=1}^k c_i\psi(p_i)\underset{\text{as } p_i\in F}{=}\sum_{i=1}^k c_ip_i$, so $\sum_{i=1}^k c_ip_i\in F$.

(e) Denote $\varphi_0=\psi$, ~~$\varphi_i=\psi_{e_i}\circ\varphi_{i-1}$ for $i=1,\cdots,n+1$ where e_i are standard bases of R^{n+1}~~
~~We prove by~~ Let $e_1\cdots e_{n+1}$ be the standard bases of R^{n+1}
If ~~all~~ $0\bar\in F$, then denote $\varphi_0=\psi_0\circ\psi$, F_0 = the set of fixed points of $\psi_0\circ\psi$. By (c) $0\in F_0$. If $0\in F$, then $\varphi_0\triangleq\psi$, $F_0=F$.
If $e_1\bar\in F_0$, then denote $\varphi_1=\psi_{e_1}\circ\varphi_0$, F_1 = the set of fixed points of φ_1.
By (c) $e_1\in F_1$, $F_0\subset F_1$ so $0\in F_1$.
The same procedure goes on, until e_{n+1}. Then $e_i\in F_{n+1}$, $i=1\cdots n+1$, $0\in F_{n+1}$.
By (d) $\sum_{i=1}^{n+1}c_ip_i\in F_{n+1}$ whenever $p_1\cdots p_{n+1}\in F$, $c_i\in R$. So $F_{n+1}=R^{n+1}$. This means φ_{n+1} is identity, i.e. there exists a $k\le n+2$, and reflections $\psi_1\cdots\psi_k$ of R^{n+1} s.t. $\psi_k\circ\cdots\circ\psi_1\circ\psi=I$. As reflections are all invertible and its inversion is itself, so $\psi=\psi_1^{-1}\circ\cdots\psi_k^{-1}=\psi_1\circ\cdots\circ\psi_k$.

Rigid motion must be injective, as if $\psi(p)=\psi(q)$, then $\|p-q\|=\|\psi(p)-\psi(q)\|=0$ so $p=q$.

22.7 (a) The set of rigid motions of R^{n+1} obviously forms a group under composition. It naturally satisfies associativity, neutral element is identity transformation, inverse element exists because rigid motions map onto R^{n+1} by ~~colla~~ corollary. Inverse is obviously rigid motion. Identity ~~belongs to~~ is a symmetry of S. For any symmetry of S ψ, as it maps onto S, it must be bijective. Its inverse is also a symmetry of S. Thus the symmetries of S form a subgroup.

(b) For any symmetry φ, suppose $\psi=\varphi_2\circ\varphi_1$ is the unique decomposition of ψ into an orthogonal transformation φ_1 followed by a translation φ_2. By definition, for any $p\in S^n$, $\varphi(p)=\varphi_1(p)+a^{\in S^n}$ (let φ_2 be translation by a). As $-p\in S^n$, $\varphi(-p)=\varphi_1(-p)+a=-\varphi_1(p)+a\in S^n$. So $\|\varphi_1(p)+a\|=1=\|-\varphi_1(p)+a\|$.

So $a\cdot\varphi_1(p)=\frac{1}{4}\left(\|\varphi_1(p)+a\|^2-\|\varphi_1(p)-a\|^2\right)=0$, so $a\cdot\varphi(p)\overset{\in S^n,\ \forall p}{=}a(\varphi_1(p)+a)=\|a\|^2$.

But as ~~$\varphi(p)$~~ φ maps onto S^n, ~~as the only possi~~ there must be a $p_0\in S^n$, s.t. $\varphi(p_0)=-a/\|a\|$, then $a\cdot\varphi(p_0)=-\|a\|\neq\|a\|^2$ unless $a=0$.

So if φ is symmetry of S^n, then φ must be an orthogonal transformation ①.

Conversely, for any orthogonal transformation ψ, if $p\in S^n$, then $\|\psi(p)\|=\|p\|=1$. So $\psi(p)\in S^n$. By Corollary, ψ maps R^{n+1} onto R^{n+1}, so for any $q\in S^n$, there must be a $p\in R^{n+1}$, s.t. $\psi(p)=q$. then $\|p\|=\|\psi(p)\|=\|q\|=1$, i.e., $p\in S^n$. Thus ψ maps S^n onto S^n. Combining ①, we prove (b)

(c) Using ~~notation~~ as in (b), let φ_2 be ~~rotation~~ translation by (a_1,a_2,a_3), and $\varphi_1=(\alpha_1,\alpha_2,\alpha_3)$

Then for any $p\in$ this cylinder C, $\varphi(p)\in C$, i.e., $(\alpha_1(p)+a_1)^2+(\alpha_2(p)+a_2)^2=a^2$ ①

As $-p$ and $\varphi(-p)\in C$, $(-\alpha_1(p)+a_1)^2+(-\alpha_2(p)+a_2)^2=a^2$ ②. ①−②: $\alpha_1(p)\cdot a_1+\alpha_2(p)\cdot a_2=0$

If φ maps C onto C, then there must be a $p_0\in C$, s.t. $\begin{pmatrix}\alpha_1(p_0)+a_1\\ \alpha_2(p_0)+a_2\end{pmatrix}=\begin{pmatrix}a_1\\ a_2\end{pmatrix}\cdot(-a/(a_1^2+a_2^2)^{1/2})$

then $\alpha_1(p_0)a_1+\alpha_2(p_0)a_2=\left[-\frac{a}{r}\begin{pmatrix}a_1\\ a_2\end{pmatrix}-\begin{pmatrix}a_1\\ a_2\end{pmatrix}\right]\cdot\begin{pmatrix}a_1\\ a_2\end{pmatrix}=-ar-r^2$, where $r=\sqrt{a_1^2+a_2^2}$.

assuming $a>0$. So $\alpha_1(p_0)a_1+\alpha_2(p_0)a_2\leq 0$ and it equals 0 iff $r=0$ i.e. $a_1=a_2=0$

Now look at restrictions on φ_1. ~~As~~ $\varphi(p)=(\alpha_1(p),\alpha_2(p),\alpha_3(p)+a_3)$

By Ex 22.3 (c), A is orthonormal.

Let the matrix of φ_1 wrt standard basis of R^3 be $A=(\beta_1,\beta_2,\beta_3)^T$, $\forall p\in C$.

~~$\varphi(p)=$~~ let $p=(p_1,p_2,p_3)$, then $\varphi(p)=\left(\sum_{k=1}^{3}\beta_{1k}p_k,\ \sum_{k=1}^{3}\beta_{2k}p_k,\ \sum_{k=1}^{3}\beta_{3k}p_k+a_3\right)$

Since p_3 can be in R. so if $\beta_{13},\beta_{23}\neq 0$, then the first two coordinates can go to infinity, rather than restricted on a circle of radius a. So $\beta_{13}=\beta_{23}=0$.

Then there is guarantee that $\left(\sum_{k=1}^{2}\beta_{1k}p_k\right)^2+\left(\sum_{k=1}^{2}\beta_{2k}p_k\right)^2=a^2$ as $\begin{pmatrix}\beta_{11}&\beta_{12}\\ \beta_{21}&\beta_{22}\end{pmatrix}$ is orthonormal ~~by Ex 22.3 (c)~~ and $\|p\|=a$. If $\beta_{33}=0$, then $\sum_{k=1}^{2}\beta_{3k}p_k+a_3$ ~~can only~~ must be bounded because p_1,p_2 are bounded ($p_1^2+p_2^2=a^2$). So $\beta_{33}\neq 0$. [This can also be seen by A being orthonormal and $\beta_{13}=\beta_{23}=0$]. But now β_{32} and β_{33} must be 0, because so far

A is like $\begin{pmatrix}\beta_{11} & \beta_{12} & 0\\ \beta_{21} & \beta_{22} & 0\\ \beta_{31} & \beta_{32} & \neq 0\end{pmatrix}$. But as $\begin{pmatrix}\beta_{11}\\ \beta_{12}\end{pmatrix} \perp \begin{pmatrix}\beta_{21}\\ \beta_{22}\end{pmatrix}$, it is impossible for $\begin{pmatrix}\beta_{31}\\ \beta_{32}\end{pmatrix}$ to be orthogonal to both $\begin{pmatrix}\beta_{11}\\ \beta_{12}\end{pmatrix}$ and $\begin{pmatrix}\beta_{21}\\ \beta_{22}\end{pmatrix}$, unless $\begin{pmatrix}\beta_{31}\\ \beta_{32}\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$. Thus $\beta_{33} = \pm 1$. In sum $A = \begin{pmatrix}\beta_{11} & \beta_{12} & 0\\ \beta_{21} & \beta_{22} & 0\\ 0 & 0 & \pm 1\end{pmatrix}$. Finally the ~~possible~~ symmetric group of cylinder $x_1^2 + x_2^2 = a^2$ in R^3 is $\varphi(P_1, P_2, P_3) = (\beta_{11}P_1 + \beta_{12}P_2,\ \beta_{21}P_1 + \beta_{22}P_2,\ \nu P_3 + a_3)$, where $\nu = 1$ or -1, $a_3 \in R$, $\begin{pmatrix}\beta_{11} & \beta_{12}\\ \beta_{21} & \beta_{22}\end{pmatrix}$ is orthonormal.

The discussion above has shown that the above conditions are both necessary and sufficient.

(d) Using same notation as in (c), $\frac{1}{a^2}(\varphi_1(P)+a_1)^2 + \frac{1}{b^2}(\varphi_2(P)+a_2)^2 + \frac{1}{c^2}(\varphi_3(P)+a_3)^2 = 1$

$\frac{1}{a^2}(-\varphi_1(P)+a_1)^2 + \frac{1}{b^2}(-\varphi_2(P)+a_2)^2 + \frac{1}{c^2}(-\varphi_3(P)+a_3)^2 = 1$, So $\frac{a_1}{a^2}\varphi_1(P) + \frac{a_2}{b^2}\varphi_2(P) + \frac{a_3}{c^2}\varphi_3(P) \geq 0$ (*)

As φ is onto, there must be a p_0 on this ellipsoid S, s.t.

$\varphi(P_0) = (\varphi_1(P_0)+a_1,\ \varphi_2(P_0)+a_2,\ \varphi_3(P_0+a_3)) = (-a_1 a, -a_2 b, -a_3 c)/r$

where $r = (a_1^2 a^2 + b^2 a_2^2 + c^2 a_3^2)^{1/2}$. Assume now $r \neq 0$.

Then $\frac{a_1}{a^2}\varphi_1(P) + \frac{a_2}{b^2}\varphi_2(P) + \frac{a_3}{c^2}\varphi_3(P) = -\frac{1}{r}\left(\frac{a_1^2}{a} + \frac{a_2^2}{b} + \frac{a_3^2}{c}\right) - \left(\frac{a_1^2}{a^2} + \frac{a_2^2}{b^2} + \frac{a_3^2}{c^2}\right) < 0$, contradicting (*)

So we must have $r = 0$, i.e. $a_1 = a_2 = a_3 = 0$.

(ii) If a, b, c are distinct, then w.l.o.g. assume $c<b$, $c<a$. Consider point $(0,0,c)$ on S. $\varphi(0,0,c) = (c\beta_{13}, c\beta_{23}, c\beta_{33})$. If it is ~~should~~ ~~which must~~ be on S, ~~thus~~ then $1 = \frac{c^2\beta_{13}^2}{a^2} + \frac{c^2\beta_{23}^2}{b^2} + \frac{c^2\beta_{33}^2}{c^2} \leq \frac{c^2}{c^2}(\beta_{13}^2 + \beta_{23}^2 + \beta_{33}^2) = 1$. ~~So the symmetry group of S is empty.~~

~~(i) If a+b=c, then same logic as above. Otherwise consider point (a,0,0) $\varphi(a,0,0) = (a\beta_{11}, a\beta_{21}, a\beta_{31})$. If it is on S, then $1 = \frac{a^2\beta_{11}^2}{a^2} + \frac{1}{b^2}a^2\beta_{21}^2 + \frac{1}{c^2}a^2\beta_{31}^2 > \frac{a^2}{a^2}(\beta_{11}^2+\beta_{21}^2+\beta_{31}^2) = 1$. So still empty is the symmetry group of S.~~

The equality holds iff $\beta_{23} = \beta_{33} = 0$ So $\beta_{13} = \pm 1$. Similarly $\beta_{21} = \beta_{31} = 0$ $\beta_{11} = \pm 1$

So A is like $\begin{pmatrix}\pm 1 & \beta_{12} & 0\\ 0 & \beta_{22} & 0\\ 0 & \beta_{32} & \pm 1\end{pmatrix}$. so $A = \begin{pmatrix}\pm 1 & 0 & 0\\ 0 & \pm 1 & 0\\ 0 & 0 & \pm 1\end{pmatrix}$. So the symmetry group of S is $\varphi(P_1, P_2, P_3) = (\delta_1 P_1, \delta_2 P_2, \delta_3 P_3)$ where $\delta_i = \pm 1$ $i = 1,2,3$.

(i) If $b = c$, $a \neq b$, then as in (ii) we have $\beta_{21} = \beta_{31} = 0$. Besides, as $(\beta_{12}b, \beta_{22}b, \beta_{32}b)$ is on S, we have $1 = \frac{b^2\beta_{12}^2}{a^2} + \frac{b^2\beta_{22}^2}{b^2} + \frac{b^2\beta_{32}^2}{c^2} \geq \frac{b^2}{b^2}(\beta_{12}^2 + \beta_{22}^2 + \beta_{32}^2) = 1$

Equality hold iff $\beta_{12} = 0$. Likewise $\beta_{13} = 0$. So A is like $\begin{pmatrix}\pm 1 & 0 & 0\\ 0 & \beta_{22} & \beta_{23}\\ 0 & \beta_{32} & \beta_{33}\end{pmatrix}$.

A is orthonormal $\Rightarrow \begin{pmatrix}\beta_{22} & \beta_{23}\\ \beta_{32} & \beta_{33}\end{pmatrix}$ is orthonormal. Conversely $\begin{pmatrix}\beta_{22} & \beta_{23}\\ \beta_{32} & \beta_{33}\end{pmatrix}$ being orthonormal is sufficient because $\varphi(P_1, P_2, P_3) = (\pm P_1, \beta_{22}P_2 + \beta_{23}P_3, \beta_{32}P_2 + \beta_{33}P_3)$ and $\frac{1}{b^2}(\beta_{22}P_2 + \beta_{23}P_3)^2 + \frac{1}{c^2}(\beta_{32}P_2 + \beta_{33}P_3)^2 = \frac{1}{b^2}[P_2^2 + P_3^2]$. So $\varphi(P_1, P_2, P_3) \in S$ and obviously $(P_2, P_3)^T \to \begin{pmatrix}\beta_{22} & \beta_{23}\\ \beta_{32} & \beta_{33}\end{pmatrix}\begin{pmatrix}P_2\\ P_3\end{pmatrix}$ is invertible and bijective from ~~S to S~~ $P_2^2 + P_3^2 = b^2\left(1 - \frac{P_1^2}{a^2}\right)$ to itself. Thus the symmetry group of S is $\varphi(P_1, P_2, P_3) = \begin{pmatrix}\pm 1 & 0 & 0\\ 0 & \beta_{22} & \beta_{23}\\ 0 & \beta_{32} & \beta_{33}\end{pmatrix}\begin{pmatrix}P_1\\ P_2\\ P_3\end{pmatrix}$, where $\begin{pmatrix}\beta_{22} & \beta_{23}\\ \beta_{32} & \beta_{33}\end{pmatrix}$ is orthonormal