Exercises in

Elementary Topics in Differential Geometry by J. A. Thorpe

1.10 $graph(f) = \{(x_1 \cdots x_{n+1}) \in R^{n+1} : (x_1 \cdots x_n) \in U,\ x_{n+1} = f(x_1 \cdots x_n)\}$

Then $graph(f)$ is a level set for $F(x_1 \cdots x_{n+1}) = 0$, where $F(x_1 \cdots x_{n+1}) = f(x_1 \cdots x_n) - x_{n+1}$

2.4 As integral curve, $\dot\alpha(t) = X(\alpha(t))$. If it crosses itself, then there exists t_1, t_2, s.t. $\alpha(t_1) = \alpha(t_2)$, $\dot\alpha(t_1) \neq \dot\alpha(t_2)$. But that isn't allowed.

2.7 (a) complete (b) incomplete say $p = (-1, 0)$ (c) complete

(d) $x_1 = tg(t+c)$, so $t \neq -c + \frac{\pi}{2}$. incomplete

2.8 Define $\tilde\beta(t) = \beta(t+t_0)$ [$t \in \{x - t_0 \mid x \in \tilde I\}$] then $\tilde\beta(0) = p$, $\dot{\tilde\beta}(t) = \dot\beta(t+t_0) = X(\beta(t+t_0)) = X(\tilde\beta(t))$ $(t \in \tilde I - t_0)$ [offset]

So $\tilde\beta(t)$ is an integral curve of X with $\tilde\beta(0) = p$. Since $\alpha(t)$ is the maximal of such curves, so for $\forall t \in \{x - t_0 \mid x \in \tilde I\}$. $\tilde\beta(t) = \alpha(t)$ i.e. $\beta(t) = \alpha(t - t_0)$ $\forall t \in \tilde I$

2.9 Define $\beta(t) \triangleq \alpha(t)$ $t \in I$, so $\beta(t_0) = \alpha(0)$. β is an integral curve of X on I

By Ex. 2.8 $\beta(t) = \alpha(t - t_0)$ i.e. $\alpha(t) = \alpha(t - t_0)$. i.e. α periodic.

(Don't worry about def. domain too much, only check & restrict in the last step)

2.10 (a) $\varphi_t(p) = p + (t, 0)$ translation. obviously one-to-one $R^2 \to R^2$

(b) $\varphi_0(p) = p + (0, 0) = p$. $\varphi_{t_1+t_2}(p) = p + (t_1 + t_2, 0) = (p + (t_2, 0)) + (t_1, 0)$

$\varphi_{-t}(p) = p + (-t, 0)$ $\varphi_t(\varphi_{-t}(p)) = p + (-t, 0) + (t, 0) = p$

2.11. (a) $\varphi_t(x_1, x_2) = (x_1 \cos t - x_2 \sin t,\ x_1 \sin t + x_2 \cos t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

rotation by t. one-to-one, additive group obviously

(b) $\varphi_t(x_1, x_2) = (x_1 e^t, x_2 e^t) = (x_1, x_2) \cdot e^t$ scaling bijection $e^{t_1 + t_2} = e^{t_1} \cdot e^{t_2}$ so additive

(c). $\varphi_t(x_1, x_2) = \frac{1}{2} \begin{pmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\det = \frac{1}{2} \cdot 4$, invertible

$= \frac{1}{2}(e^t + e^{-t}) \begin{pmatrix} 1 & \tanh(t) \\ \tanh(t) & 1 \end{pmatrix}$ Use $\tanh(t_1 + t_2) = \frac{\tanh(t_1) + \tanh(t_2)}{1 - \tanh(t_1) \cdot \tanh(t_2)}$

2.12 Suppose $\beta(t)$ is the integral curve of X with $\beta(0) = \varphi_{t_2}(p)$. so $\alpha(0) = p$. $\alpha(t_2) = \beta(0)$

By using Ex. 2.8 (now the α here is the β is Ex. 2.8), $\beta(t) = \alpha(t + t_2)$

$\varphi_{t_1}(\varphi_{t_2}(p)) = \beta(t_1) = \alpha(t_1 + t_2) = \varphi_{t_1+t_2}(p)$, $\varphi_{-t_2}(\varphi_{t_2}(p)) = \beta(-t_2) = \alpha(0) = p$ $\forall t_2$

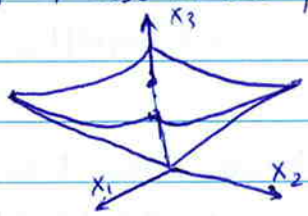
3.1 $n=1$ $f = x_1^2 - x_2^2$ $f^{-1}(-1)$ is $x_1^2 = x_2^2+1$ $\nabla f = (2x_1, -2x_2)$ so $\nabla f \neq 0$, no such p

$f^{-1}(1)$ also doesn't have such p. $f^{-1}(0)$. $\nabla f(0,0) = (0,0)$. $f^{-1}(0)$ is $x_1 = \pm x_2$

Its tangent space is $\{\lambda(1,1), \lambda(1,-1) \mid \lambda \in R\} \neq [\nabla f(0,0)]^{\perp} = R^2$

$n=2$ $f = x_1^2 + x_2^2 - x_3^2$ $f^{-1}(-1)$ is $x_1^2 + x_2^2 = x_3^2 + 1$. $\nabla f \neq 0$ no such p. $f^{-1}(1)$ also no such p

$f^{-1}(0)$: $x_3^2 = x_1^2 + x_2^2$ at $p = (0,0)$, the tangent space



at $(0,0,0)$ is all vectors V (x_1, x_2, x_3) where V is 45° to x_3 axis

$|V \cdot (0,0,1) / \|V\|| = \frac{\sqrt{2}}{2}$ i.e. $x_3^2 = x_1^2 + x_2^2 \neq [\nabla f(0)]^{\perp}$

3.2 (a) the example in 3.1 with $n=1$, $c=0$. $f^{-1}(0)$: $x_1 = \pm x_2$ $(1,1), (1,-1) \in S$, $(1,0) \notin S$

(b) $f(x_1 \cdots x_{n+1}) = c$. $S = f^{-1}(c)$, tangent space $= R_p^{n+1}$

3.4 $f \circ \alpha = c \Leftrightarrow \frac{d(f(\alpha(t)))}{dt} = 0 \Leftrightarrow \nabla f(\alpha(t)) \cdot \dot{\alpha}(t) = 0 \Leftrightarrow \dot{\alpha} \perp \nabla f(\alpha)$ $\forall t$.

3.5 α is integral curve of $\nabla f \Rightarrow \dot{\alpha} = \nabla f(\alpha)$

(a) $\frac{d}{dt} f(\alpha(t)) = \nabla f(\alpha(t)) \cdot \dot{\alpha}(t) = \|\nabla f(\alpha(t))\|^2$

(b) $\frac{d}{dt} f(\beta(s_0)) = \nabla f(\beta(s_0)) \cdot \dot{\beta}(s_0) = \nabla f(\alpha(t_0)) \cdot \dot{\beta}(s_0)$. As $\|\dot{\beta}(s_0)\| = \|\dot{\alpha}(t_0)\|$

it is maximized when $\dot{\beta}(s_0) = \dot{\alpha}(t_0) = \nabla f(\alpha(t_0))$, then

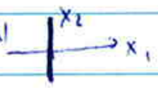
$\frac{d}{dt} f(\beta(s_0)) = \|\nabla f(\alpha(t_0))\|^2 = \frac{d}{dt} f(\alpha(t_0))$ by (a)

4.3 Consider $S = f^{-1}(c)$. $\forall p \in S$. p is an extreme point of g on S.

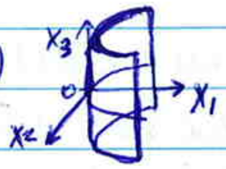
By Lagrange Theorem, $\nabla g(p) = \lambda \cdot \nabla f(p)$ $\forall p \in S$. $\lambda \neq 0$ because $\nabla g(p) \neq 0$ for all $p \in S$

4.4 See http://users.rsise.anu.edu.au/~xzhang/dg_thorpe/monkey.jpg

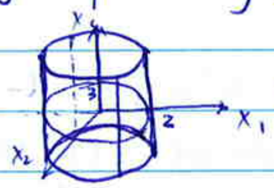
4.5 (a)



the x_2-axis (b)



(c)



ellipse on $x_3 = 0$

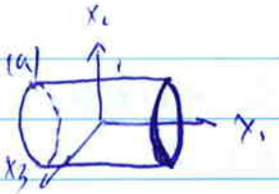
4.6

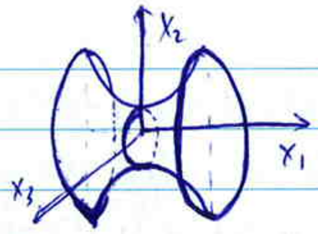
4.7 $g(x_1, x_2, x_3) = f(x_1, (x_2^2 + x_3^2)^{1/2})$, then $\frac{\partial g}{\partial x_1} = \frac{\partial f}{\partial x_1}$. Denote $u = (x_2^2 + x_3^2)^{1/2}$

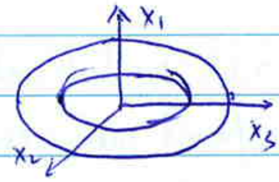
then $\frac{\partial g}{\partial x_2} = \frac{\partial f}{\partial u} x_2 (x_2^2 + x_3^2)^{-1/2}$, $\frac{\partial g}{\partial x_3} = \frac{\partial f}{\partial u} x_3 (x_2^2 + x_3^2)^{-1/2}$. If $\nabla g(p) = 0$, then

$\frac{\partial g}{\partial x_2} = \frac{\partial g}{\partial x_3} = 0$, i.e. $0 = (\frac{\partial g}{\partial x_2})^2 + (\frac{\partial g}{\partial x_3})^2 = [\frac{\partial f}{\partial u}]^2 = 0$. So $\frac{\partial f}{\partial u} = 0$. Besides $\frac{\partial f}{\partial x_1} = \frac{\partial g}{\partial x_1} = 0$

So $\nabla f = 0$ at p, which contradicts with the fact that ∇f ~~is a surface~~ $\neq 0$

4.8 (a) 

(b) 

(c) 

ref Fig 14.5 Pg. 112 (change axis)

4.9 $f(x) = x_3^2 + x_4^2 - 1$ $S = f^{-1}(0)$. $\nabla f = (0, 0, 2x_3, 2x_4)$ If $\nabla f = 0 \Rightarrow x_3 = x_4 = 0 \Rightarrow$ not on S

4.10 $g(x_1, x_2, x_3, x_4) = f(x_1, x_2, (x_3^2 + x_4^2)^{1/2})$

4.11 By Lagrange Thm, $\nabla g = \lambda \nabla f$. $\Rightarrow \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = \lambda \begin{pmatrix} 2ax_1 + 2bx_2 \\ 2bx_1 + 2cx_2 \end{pmatrix} \Rightarrow \lambda \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Since $ac - b^2 > 0 \Rightarrow \det \begin{pmatrix} a & b \\ b & c \end{pmatrix} \neq 0 \Rightarrow \lambda \neq 0 \Rightarrow \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

At that point $g = (x_1, x_2)^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda (x_1, x_2) \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda$. Note λ^{-1} is eigenvalue of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

4.12 $g = x^T A x$ $\nabla g = 2Ax$ $f = \sum_{i=1}^{n} x_i^2$. $\nabla f = 2x$ $\nabla g = \lambda \nabla f \Rightarrow Ax = \lambda x$

$g(x) = \lambda x^T x = \lambda$ the eigenvalue of A

4.13 By Lagrange Thm, $\lambda \nabla f(p) = \nabla g(p)$. As $\nabla g(p) \neq 0$ $\lambda \neq 0$ $\forall v$: $v \cdot \nabla g(p) = 0 \Leftrightarrow v \cdot \nabla f(p) = 0$

So tangent space of g through p is equal to tangent space of f through p

4.14 Let $g = \|x - p_0\|^2$. $S = f^{-1}(c)$. Since p is an extreme point of g on S

$\nabla g(p) = \lambda \nabla f(p)$ But $\nabla g(p) = 2(p - p_0)$. So $(p, p - p_0) \perp S_p$.

4.15 $\nabla \det(X) = \frac{1}{\det(X)} (X^{-1})^T$ So $\nabla \det(X) = 0$ is impossible.

4.16 (a) $\nabla \det(X) = \frac{1}{1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So $\left\langle \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle_F = 0 \Rightarrow x_1 + x_4 = 0$

(b) $\nabla \det(g) = \frac{1}{1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$ So $SL(2)_g = \{ (p, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) : a - b - c + 2d = 0 \}$.

4.17 (a) The proof in 4.15 is independent of dimension

(b) $\nabla \det(p) = I$. So $SL(3)_p = \{ (p, M) \mid M \in R^{3 \times 3}, tr(M) = 0 \}$

5.1 ~~Only need to prove every point is connected to origin $(0,0,1)$ $\forall x$, define~~

$\forall x_1, x_2$, consider parametrized curve $\alpha(t) = x_1 \cos t + u \sin t$ ~~where $u = \frac{p_1 - x_1 \cos\theta}{\sin\theta}$~~

~~then $\alpha(0) = x_1$~~, where $u = \frac{x_2 - x_1 \cos\theta}{\sin\theta}$, here $\theta = \cos^{-1}(x_1 \cdot x_2)$ if $\sin\theta \neq 0$ ~~here $\theta = \arccos(x$ (if $\sin\theta \neq 0)$~~

then $\alpha(0) = x_1$ $\alpha(\theta) = x_1 \cos\theta + \frac{x_2 - x_1 \cos\theta}{\sin\theta} \sin\theta = x_2$,

$\|u\| = \frac{1}{\sin^2\theta} (1 + \cos^2\theta - 2\cos\theta \cdot \langle x_1, x_2 \rangle) = \frac{1}{\sin^2\theta} (1 + \cos^2\theta - 2\cos^2\theta) = 1$, $\langle u, x_1 \rangle = \frac{\langle x_1, x_2 \rangle - \cos\theta}{\sin\theta} = \frac{\cos\theta - \cos\theta}{\sin\theta} = 0$

So $\|\alpha(t)\| = \|x_1\|^2\cos t + \|u\|^2\sin t + \langle x_1, u\rangle \sin t\cos t = \cos^2 t + \sin^2 t = 1$ So $\alpha(t) \in S$.

So far, we've found the curve. If $\sin\theta = 0$. Then $x_1 = x_2$ or $x_1 = -x_2$

If $x_1 = x_2$, done. If $x_1 = -x_2$. then find a u, s.t. $\|u\| = 1$ and $\langle x_1, u\rangle = 0$ and $\alpha(t) = x_1\cos t + u\sin t$

$\|\alpha(t)\| = 1$, $\alpha(0) = x_1$ $\alpha(\pi) = -x_1 = x_2$ □ QED.

Note. A easier way is by using polar angular axis.

Let $x_1 = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \cdots, \sin\theta_1\cdots\sin\theta_{n-1}\cos\theta_n, \sin\theta_1\cdots\sin\theta_n)$

$x_2 = (\cos\theta_1', \sin\theta_1'\cos\theta_2', \sin\theta_1'\sin\theta_2'\cos\theta_3', \cdots, \sin\theta_1'\cdots\sin\theta_{n-1}'\cos\theta_n', \sin\theta_1'\cdots\sin\theta_n')$

So we just need to find a continuous curve from $(\theta_1, \cdots, \theta_n) \to (\theta_1', \cdots \theta_n')$ in $[0, 2\pi)^n$

But $[0, 2\pi)^n$ is convex set, so just easily find $\beta(t)$. s.t. $\beta(t) \in [0, 2\pi)^n$

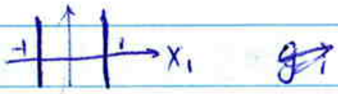
$\beta(0) = (\theta_1, \cdots, \theta_n)$ $\beta(t_0) = (\theta_1', \cdots, \theta_n')$. Then define

$\alpha(t) = (\cos\beta_1(t), \sin\beta_1(t)\cos\beta_2(t), \cdots, \sin\beta_1(t)\cdots\sin\beta_{n-1}(t)\cos\beta_n(t), \sin\beta_1(t)\cdots\sin\beta_n(t))$

5.2 If there exists $p, q \in S$, s.t. $g(p) = 1$, $g(q) = -1$. then as S is connected, there exists a continuous map $\alpha: [a,b] \to S$, s.t $\alpha(a) = p$, $\alpha(b) = q$

As $g\circ\alpha$ is continuous, $g(\alpha(a)) = 1$. $g(\alpha(b)) = -1$, So there exists $c \in (a, b)$, s.t. $g(\alpha(c)) = 0$

But by definition of α, $\alpha(c) \in S$ which contradicts with $g(x) = \pm 1$ for $\forall x \in S$.

5.3 1-surface: $f(x_1, x_2) = (x_1 - 1)(x_1 + 1)$ 

Define $g(x_1, x_2) = \begin{cases} -1 & x_1 \in (-3/2, -1/2) \\ 1 & x_1 \in (1/2, 3/2) \end{cases}$ So g is smooth on S, but g is not constant

5.4 $N_1(p)$ and $N_2(p)$ are both smooth. $\|\pm p/r\| = 1$, $\pm p/r \in S_p^\perp$ $\nabla(\sum_{i=1}^n x_i^2) = 2(x_1, \cdots x_n)^T$

5.5 (a)  rotate counterclock-wise by $\pi/2$

(b) $R_\theta((v, 0)) = \cos\theta \cdot (v, 0) + \sin\theta \cdot (0, 0, 1) \times (v, 0) = (v', 0)$ where $v' = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} v$

which is counterclock-wise rotation with angle θ.

(c) $\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 > 0$ So right-handed

5.6 Let θ denote the angle measured counterclock-wise from $(p, 1, 0)$ to the orientation direction $N(p)$, so that $N(p) = (p, \cos\theta, \sin\theta)$ So the positive tangent direction is $(\cos(\theta - \frac{\pi}{2}), \sin(\theta - \frac{\pi}{2})) = (\sin\theta, -\cos\theta)$. v is tangent to C at p, So $v/\|v\| = \pm(\sin\theta, -\cos\theta)$

But if $v/\|v\| = -(\sin\theta, -\cos\theta)$ then $\det\begin{pmatrix} v/\|v\| \\ N(p) \end{pmatrix} = \det\begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} = -1$. which means inconsistent

So positive tangent $\iff$ consistent

5.7 (a)(b) just write out (c) take $u = (1,0,0), (0,1,0), (0,0,1)$ then get it

5.8 (a) consistent $\Leftrightarrow \det\begin{pmatrix} v \\ w \\ N(p)\end{pmatrix} > 0 \Leftrightarrow v\cdot(w\times N(p)) > 0 \Leftrightarrow N(p)\cdot(v\times w) > 0$

(b) Denote $\hat{x} \triangleq x/\|x\|$, consistent $\Leftrightarrow \hat{w}\cdot(N(p)\times\hat{v}) > 0$

~~[As {v,w} is a basis of S_p As $\|\hat w\|=1$ So there must exist θ. $\hat{w} = \cos\theta\,\hat{v} + \sin\theta\, N(p)\times\hat{v}$~~

(Proof) As $N(p)\cdot(N(p)\times\hat{v}) = \det\begin{pmatrix} N(p) \\ N(p) \\ \hat v\end{pmatrix} = 0$ So $N(p)\times\hat{v} \in S_p$.

$\hat{v}\cdot(N(p)\times\hat{v}) = \det\begin{pmatrix} \hat v \\ N(p) \\ \hat v\end{pmatrix} = 0$. So $\{N(p)\times\hat{v}, \hat{v}\}$ is an orthornormal basis of S_p

As $\|\hat{w}\| = 1$. So there exists θ s.t. $\hat{w} = \cos\theta\cdot\hat{v} + \sin\theta\cdot N(p)\times\hat{v}$

So $\hat{w}\cdot(N(p)\times\hat{v}) = \sin\theta$

So $\theta \in (0,\pi) \Leftrightarrow \hat{w}\cdot(N(p)\times\hat{v}) > 0 \Leftrightarrow \{v,w\}$ is consistent with N

5.9 (a) take $u = (1,0,0,0), (0,1,0,0) \cdots (0001)$ (b) just check

5.10 (a) $\det\begin{pmatrix} v_1 \\ \vdots \\ v_n \\ N\end{pmatrix} < 0 \Leftrightarrow \det\begin{pmatrix} v_1 \\ \vdots \\ v_n \\ -N\end{pmatrix} > 0$

(b) Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n\end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ \vdots \\ w_n\end{pmatrix}$ $\begin{pmatrix} W \\ N\end{pmatrix} = \begin{pmatrix} AV \\ N\end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} V \\ N\end{pmatrix}$ where $W = \begin{pmatrix} w_1 \\ \vdots \\ w_n\end{pmatrix}$, $V = \begin{pmatrix} v_1 \\ \vdots \\ v_n\end{pmatrix}$

So $\det\begin{pmatrix} W \\ N\end{pmatrix} = \det A\cdot\det\begin{pmatrix} V \\ N\end{pmatrix}$, thus consistency of W with N is identical to the consistency of V with N iff $\det A > 0$

6.1 ~~N(S) = {v | ‖v‖ ...~~ $n=1$ $N(S) = \{(0,1),(0,-1)\}$; $n=2$ $N(S) = \{(0,x_2,x_3) \mid x_2^2 + x_3^2 = 1\} \in \mathbb{R}^3$

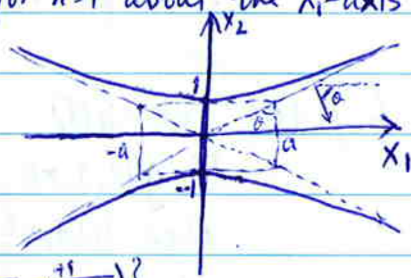
6.2 $n=1$ $N(S) = \{(\frac{-\sqrt2}{2}, \frac{\sqrt2}{2}), (\frac{-\sqrt2}{2}, \frac{-\sqrt2}{2})\}$; $n=2$ $N(S) = \{(\frac{-\sqrt2}{2}, u, v) \mid u^2+v^2 = \frac12\}$

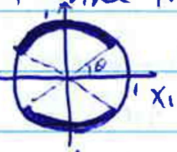
6.3 $n=1$ $N(S) = \{(x_1,x_2) \mid x_1^2 + x_2^2 = 1\}$; $n=2$. $N(S) = \{(x_1\ x_2\ x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$

6.4 $n=1$ $N(S) = \{(x_1,x_2) \mid x_1^2 + x_2^2 = 1, x_1 < 0\}$; $n=2$ $N(S) = \{(x_1,x_2,x_3) \mid \sum_{i=1}^{3} x_i^2 = 1, x_1 < 0\}$

6.5 We only ~~heed~~ need to analyze $n=1$, the cases for $n \geq 2$ can be ~~derived by~~ viewed as the surface of revolution obtained by rotating the curve for $n=1$ about the x_1-axis then about (x_1,x_2)-plane ~~axis~~, then about (x_1,x_2,x_3) ...

For $n=1$ $-\frac{x_1^2}{a^2} + x_2^2 = 1$ like the right figure.



The spherical image is 

$\theta = tg^{-1} a$. or formally $\{(x_1,x_2) \in S^1 \mid x_1 \in (\frac{-1}{\sqrt{a^2+1}}, \frac{1}{\sqrt{a^2+1}})\}$

For $n \geq 2$. The spherical image is $\{(x_1, \cdots x_{n+1}) \in S^n \mid x_1 \in (\frac{-1}{\sqrt{a^2+1}}, \frac{+1}{\sqrt{a^2+1}})\}$

When $a \to \infty$, it shrinks to a narrow band.

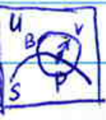
when $a \to 0$, it extends to the whole S^n

6.6 Obvious ?

6.7 "if part": Suppose the orientation at p is $N(p)$. Since $\alpha(t) = p + ta \in S$ for all $t \in I$, so $\dot{\alpha}(0) = a \in S_p$. So $a \cdot N(p) = 0$ which is true for any $p \in S$.

"only if" part: Consider the constant vector field $X(q) = (q, a)$. It is a tangent field on S because at $\forall p$. $N(p) \cdot a = 0$. Now $\alpha(t) = p + at$ is an integral curve of X and $\alpha(0) = p \in S$. Then by the corollary to Theorem 1, Chapter 5, $\alpha(t) \in S$ for all $t \in I$ where I is the interval on which $\alpha(t)$ is defined.

6.8 Suppose $N(S) = \{v\}$. Let B be an open ball contained in U (S is a level set on U) and $p \in S \cap B$. Then for $\forall x_0 \in B$ which satisfies $(x_0 - p) \cdot v = 0$, we construct a constant vector field $W(q) = (q, x_0 - p)$, ~~which is the those~~ Since $N(S) = \{v\}$, the restriction of $W(q)$ on U is a tangent vector field on S. As B is open, there is $\alpha(t) = p + (x_0 - p)t$ ($(-\varepsilon, 1+\varepsilon) \to B$), an integral curve of W, such that $\alpha(0) \in S$. Thus by corollary to Thm 1, ch 5, $\alpha(t) \in S$, and specifically $\alpha(1) = x_0 \in S$. Therefore $\{x \in \mathbb{R}^{n+1} : x \cdot v = p \cdot v\} \cap B \subseteq S$.



Next, suppose a new $\alpha : [a, b] \to S$ is a continuous parametrized curve and $\alpha(t) \in B'$ (a new open set) for $t_1 \le t \le t_2$. If $\alpha(t_1) \cdot v < \alpha(t_2) \cdot v$, then for any $b \in (\alpha(t_1) \cdot v, \alpha(t_2) \cdot v)$, due to $\alpha(t)$ being continuous, there exists $t_3 \in (t_1, t_2)$ s.t. $\alpha(t_3) \cdot v = b$. Since $\alpha(t_3) \in S \cap B$, by above argument, we have $\{x \in \mathbb{R}^{n+1} : x \cdot v = \alpha(t_3) \cdot v = b\} \cap B \subseteq S$. Therefore $\{x \in B \mid x \cdot v \in (\alpha(t_1) \cdot v, \alpha(t_2) \cdot v)\} \subseteq S$. But the left hand set is an open set and therefore $N(S) = S^n$ (because S contains an open set), contradicting with $N(S) = \{v\}$. So $\alpha(t_1) \cdot v \geq \alpha(t_2) \cdot v$. Likewise, $\alpha(t_1) v \leq \alpha(t_2) \cdot v$.

So $\alpha(t_1) \cdot v = \alpha(t_2) \cdot v$. Since S is connected, any two points on S p, q can be connected by a continuous parametrized curve, and one can find an open set B s.t. $p, q \in S \cap B$, therefore $p \cdot v = q \cdot v$, i.e., all points in S lie on the same plane (or part of a plane).

6.9 (a) Let $g(t) = f(\alpha(t))$. So now we have $g(t_1) = g(t_2) = c$, $g(t) \neq c$ for all $t \in (t_1, t_2)$.

If $g'(t_1) > 0$, $g'(t_2) > 0$. Then there exists $\varepsilon_1 > 0$, s.t. $g'(t) > 0$, $t \in [t_1, \varepsilon_1 + t_1)$

then $g(t_1 + \frac{\varepsilon_1}{2}) - g(t_1) = g'(t_1 + \xi_1) \cdot \frac{\varepsilon_1}{2}$ where $\xi_1 \in [0, \frac{\varepsilon_1}{2}]$. So $g'(t_1 + \xi_1) > 0$,

thus $g(t_1 + \frac{\varepsilon_1}{2}) > g(t_1) = c$. There also exists $\varepsilon_2 > 0$ s.t. $g'(t) > 0$, $t \in (t_2 - \varepsilon_2, t_2]$

then $g(t_2 - \frac{\varepsilon_2}{2}) - g(t_2) = -g'(t_2 - \xi_2) \frac{\varepsilon_2}{2}$, where $\xi_2 \in [0, \frac{\varepsilon_2}{2}]$ so $g'(t_2 - \xi_2) > 0$

thus $g(t_2 - \frac{\varepsilon_2}{2}) < g(t_2) = c$. Then as g is continuous there exists $t \in (t_1 + \frac{\varepsilon_1}{2}, t_2 - \frac{\varepsilon_2}{2}) \subset (t_1, t_2)$

s.t. $g(t) = c$, contradiction!

one can choose small enough $\varepsilon_1, \varepsilon_2$, s.t. $t_1 + \frac{\varepsilon_1}{2} < t_2 - \frac{\varepsilon_2}{2}$

~~(a)~~ if $g(t_1)<0$, $g(t_2)<0$, same contradiction occurs. So $g(t_1)g(t_2)<0$

(b) If α crosses S for an odd number of times $t_1 \cdots t_n$ then by (a) $g'(t_1)g'(t_n)>0$. Without loss of generality, suppose $g'(t_1)>0$, $g'(t_n)>0$.

Since $g(t_1)=g(t_n)=c$ (and t_1, t_n are two extreme times), So $g(t)<c$ for all $t<t_1$; $g(t)>c$ for all $t>t_n$

However as S is compact ~~and α goes to ∞ in both directions we can~~ and $f: R^{n+1} \to R$ ~~there is a bar~~ Suppose S is strictly contained in sphere S': $||x||^2 = r^2$, ~~then pick any $p \in S$ and consider $S_1 f^{-1}(f(p))$~~. Since α goes to ∞ in both directions

There must be t_0, t_{n+1} with $t_0<t_1$, $t_{n+1}>t_n$, such that $\alpha(t_0)$ and $\alpha(t_{n+1}) \in S'$

~~As $f(\alpha(t_0))<c$, $f(\alpha(t_{n+1}))>c$ and f is continuous on S'~~, So $f(\alpha(t_0))<c$, $f(\alpha(t_{n+1}))>c$

As S' is connected (see Ex. 5.1), there is a continuous parametrized curve $\beta(t) \in S'$, s.t. $\beta(t^1)=\alpha(t_0)$, $\beta(t^2)=\alpha(t_{n+1})$. As f, β are continuous on S', there must be a $t^3 \in (t^1, t^2)$ s.t. $f(\beta(t^3))=c$.

But $\beta(t^3) \in S'$, So $\beta(t^3) \bar{\in} S$. This is contradiction!

6.10 (a) ~~$\forall \beta(t)$~~. Since $\beta(a) \in O(S)$, there exists a continuous map $\alpha: [0,+\infty) \to R^{n+1}-S$ s.t. $\alpha(0)=\beta(a)$, $\lim_{t\to\infty}||\alpha(t)|| = \infty$.

For $\forall \beta(t_0)$. Construct curve $\gamma(t) = \begin{cases} \beta(t_0-t) & t \in [0, t_0-a] \\ \alpha(t-t_0+a) & t \in (t_0-a, +\infty) \end{cases}$

then $\gamma(t)$ is continuous from $[0,+\infty) \to R^{n+1}-S$. $\gamma(0)=\beta(t_0)$, $\gamma(+\infty)=\alpha(+\infty)=\infty$

t_0 is arbitrary so $\beta(t) \in O(S)$ for all $t \in [a,b]$

(b) ~~Open set. $\forall p$~~ (1) Non-empty. As S is a compact n-surface, ~~there~~ we can find a n-sphere with a large enough radius r, which strictly subsumes S then pick one point on the n-sphere, p, construct continuous map $\alpha(t) = pt+p$. So $\alpha(0)=p$. $\forall t>0$. $||\alpha(t)|| = (t+1)r > r$

So $\alpha(t) \in R^{n+1}-S$, $\lim_{t\to\infty}||\alpha(t)||=\infty$. So $p \in O(S)$

(2) open set $\forall p \in O(S)$, $p \in R^{n+1}-S$, as $R^{n+1}-S$ is open (due to $S=f^{-1}(c)$ is n-surface and by definition f is smooth). So there exists an ε-ball around p, (p,ε), such that $\forall x \in (p,\varepsilon)$ satisfy $x \in R^{n+1}-S$

we can easily construct a continuous map from p to x, By (a), $x \in O(S)$, $\forall x \in (p,\varepsilon)$

(3) connected: $\forall p, q \in O(S)$, Suppose there is a n-sphere S_1 with radius r ~~s.t.~~ such that p, q, S are all contained in it (S compact, $r>||p||$, $r>||q||$).

As $p \in O(S)$ there is a continuous map $\alpha_1: [0,+\infty) \to R^{n+1}-S$, $\alpha_1(0)=p$, $\lim_{t\to\infty}||\alpha_1(t)||=\infty$.

Suppose ~~$\alpha(t)$~~ $||\alpha_1(t_1)|| = r$ (i.e. $\alpha_1(t_1) \in S_1$). Likewise, we define $\alpha_2(t)$ and t_2

As S_1 is connected and $S_1 \subset R^{n+1}-S$, there's a curve α_3 on S_1, s.t. $\alpha_3(a)=\alpha_1(t_1)$, ~~$\alpha_3(b)$~~

$\alpha_3(b)=\alpha_2(t_2)$. $\alpha_3(t)\in O(S)$ by (a). So now construct continuous curve from p to q in O(S):

$$\gamma(t)=\begin{cases}\alpha_1(t) & t\in[0,t_1]\\ \alpha_3(t-t_1+a) & t\in[t_1,t_1+b-a]\\ \alpha_2(t_2-t+t_1+b-a) & t\in(t_1+b-a,\ t_1+b-a+t_2)\end{cases}$$

7.2 $\|\dot\alpha(t)\|=$ constant $\Rightarrow \frac{d}{dt}\dot\alpha(t)\cdot\dot\alpha(t)=2\ddot\alpha(t)\cdot\dot\alpha(t)=0$, i.e. $\ddot\alpha(t)\perp\dot\alpha(t)$

7.3 Let $s(t)=\int_{t_0}^{t}\|\dot\alpha(t)\|dt$. As $\dot\alpha(t)\neq 0$, so $s(t)$ monotonic increasing so $s(t)$ is invertible. Let $h=s^{-1}$. h is onto by definition $h'=\frac{1}{s'}=\|\dot\alpha(h(t))\|^{-1}>0$

$\dot\beta=\dot\alpha(h(t))\cdot h'(t)=\dot\alpha(h(t))/\|\dot\alpha(h(t))\|$ so β is unit speed

7.5. "if part" is by Example 2 in this chapter

"only if". $\alpha(0)=(r\cos b, r\sin b, d)$, which has covered all possible points on cylinder

$\dot\alpha(0)=(-ra\sin b, ra\cos b, c)$. $N_{\alpha(0)}=\pm(\cos b,\sin b, 0)$.

So $\dot\alpha(0)$ has covered all possible initial velocity in $S_{\alpha(0)}$

As geodesic is uniquely determined by initial position and initial velocity these are all possible geodesics on cylinder S.

Another proof is by looking at (G) on page 41. $N(x,y,z)=(x,y,0)$

7.6 "if part" is covered by Example 3 in this chapter

"only if" $\alpha(0)=e_1$, $\dot\alpha(0)=ae_2$. Since $e_2\in S_{e_1}$. a allows all norm of velocity

e_1 allows all possible initial position, $\dot\alpha(0)$ allows all possible initial velocity

due to uniqueness of geodesic by initial position and velocity, these are all possible geodesics on unit n-sphere.

7.7 "if part": $\dot\beta(t)=\dot\alpha(h(t))h'(t)=\dot\alpha(at+b)\cdot a$; $\ddot\beta(t)=a^2\ddot\alpha(at+b)$. As $\alpha(t)$ is geodesic so $\ddot\alpha(t)\in S^{\perp}_{\alpha(t)}\ \forall t$. So $\ddot\beta(t)\in S^{\perp}_{\alpha(at+b)}=S^{\perp}_{\beta(t)}$ So β is geodesic

"only if": $\ddot\beta(t)=\ddot\alpha(h(t))\cdot(h'(t))^2+\dot\alpha(h(t))h''(t)$ if β is geodesic, $\ddot\beta(t)\in S^{\perp}_{\beta(t)}=S^{\perp}_{\alpha(h(t))}\ni\ddot\alpha(h(t))$

So $\ddot\beta(t)$ and $\ddot\alpha(h(t))$ are parallel → generally, $\dot\alpha$ and $\ddot\alpha$ are not parallel, and $h''(t)$, $h'(t)$ are scalar

So we must require $h''(t)=0$ (E.g. $\alpha(t)=\hat e_1\cos t+\hat e_2\sin t$ $\dot\alpha(t)=-\hat e_1\sin t+\hat e_2\cos t$

$\ddot\alpha(t)=-\hat e_1\cos t-\hat e_2\sin t$, $\theta_{\dot\alpha,\ddot\alpha}=0$. So $\dot\alpha$ and $\ddot\alpha$ are never parallel).

So $h(t)=at+b$. We can't see why we need $a>0$. Since at least when $a=0$, β is still geodesic

7.8 (a) $\dot\alpha_\theta(t) = (\dot x_1(t), \dot x_2(t)\cos\theta, \dot x_2(t)\sin\theta)$

$\dot\beta_t(\theta) = (0, -x_2(t)\sin\theta, x_2(t)\cos\theta)$

$\dot\alpha_\theta(t)\cdot\dot\beta_t(\theta) = 0$

(b) $\ddot\alpha_\theta(t) = (\ddot x_1(t), \ddot x_2(t)\cos\theta, \ddot x_2(t)\sin\theta)$

~~Sp~~ $N(p) = \pm ?$ hard to write. So must find another way

Notice that $\dot\alpha_\theta(t) \in S_p$. $\dot\beta_t(\theta) \in S_p$ by definition because $\alpha_\theta(t), \beta_t(\theta)$ are both on S.

by (a) $\dot\alpha_\theta(t) \perp \dot\beta_t(\theta)$. So $\dot\alpha_\theta(t), \dot\beta_t(\theta)$ form a basis of S_p $(p = \alpha_\theta(t))$

So one only needs to check that $\ddot\alpha_\theta(t)$ is orthogonal to $\dot\alpha_\theta(t)$ and $\dot\beta_t(\theta)$

~~$\dot\alpha_\theta(t)\cdot\ddot\alpha_\theta(t) = \ddot x_1(t)\dot x_1(t) + \ddot x_2(t)\dot x_2(t)$~~. As $\alpha(t) = (x_1(t), x_2(t))$ has constant speed, by Ex 7.2 $\dot\alpha_\theta(t) \perp \ddot\alpha_\theta(t)$. $\ddot\alpha_\theta(t) \perp \dot\beta_t(\theta)$ is easy to check.

(c) $\ddot\beta_t(\theta) = (0, -x_2(t)\cos\theta, -x_2(t)\sin\theta)$, obviously $\ddot\beta_t(\theta) \perp \dot\beta_t(\theta)$

$\ddot\beta_t(\theta) \perp \dot\alpha_\theta(t) \Leftrightarrow x_2(t)\cdot\dot x_2(t) = 0$ Since $x_2(t) > 0$ $\dot x_2(t) = 0 \Leftrightarrow x_2'(t)/x_1'(t)\big|_{\alpha(t)} = 0$

7.9 First check $\beta(t) = \alpha(ct)$ is a maximal ~~d~~ geodesic with initial velocity cv: $\beta(0) = \alpha(0)$

$\dot\alpha(ct) = c\cdot\dot\alpha(t)$ ~~= cv~~. So $\dot\alpha(ct)\big|_{t=0} = c\cdot\dot\alpha(t)\big|_{t=0} = cv$.

$\ddot\beta(t) = c^2\ddot\alpha(ct)$. As α is geodesic, so $\ddot\alpha(t) \in S^\perp_{\alpha(t)}$. So $\ddot\beta(t) \in S^\perp_{\alpha(ct)} = S^\perp_{\beta(t)}$

So $\beta(t)$ is geodesic. ~~I is easily~~ Since the geodesic with given initial position and velocity ~~given~~ is unique, $\beta(t)$ is ~~what w~~ the maximal geodesic in S with initial velocity cv.

The domain I can be easily taken care of.

7.10 Define $\gamma(t) = \beta(t+t_0)$, then $\gamma(0) = \beta(t_0) = p$, $\dot\gamma(0) = \dot\beta(t_0) = v$. So if $\nu(t)$ is geodesic, then by uniqueness theorem, $\gamma(t) = \alpha(t)$, i.e. $\beta(t+t_0) = \alpha(t)$, i.e. $\beta(t) = \alpha(t-t_0)$

$\tilde I$ is taken care of because α is maximal

7.11 Let $\nu(t) = \beta(t)$. $\nu(t_0) = \beta(t_0) = \beta(0)$ ~~= v(t_0)~~, $\dot\nu(t_0) = \dot\beta(t_0) = \dot\beta(0)$. So by Ex 7.10

$\nu(t) = \beta(t-t_0)$ i.e. $\beta(t) = \beta(t-t_0)$ i.e. $\beta(t+t_0) = \beta(t)$

7.12 (a) complete by Example 3

(b) incomplete $\alpha(t) = (1,0,\cdots,0)\cos t + (0,\cdots,0,1)\sin t$ is geodesic $t \in (-\frac{3\pi}{2}, \frac{\pi}{2})$ but $t \neq \frac{\pi}{2} + 2k\pi$ $k \in \mathbb{Z}$

(c) incomplete $\alpha(t) = (0,1,1) - (0,1,1)t$ $t \neq 1$

(d) complete by Example 2

(e) incomplete $\alpha(t) = (0,1,0)\cos t + (1,0,0)\sin t$ $t \neq \frac{\pi}{2} \pm 2k\pi$ $k \in \mathbb{Z}$

8.1 (b) $(fX)' = (\dot{f}X) - [(\dot{f}X)\cdot N(\alpha(t))]N(\alpha(t))$

$= \dot{f}X + f\dot{X} - [(\dot{f}X + f\dot{X})\cdot N(\alpha(t))]N(\alpha(t))$ (as $X\cdot N(\alpha(t)) = 0$ by X being tangent to S)

$= \dot{f}X + f\dot{X} - f[\dot{X}\cdot N(\alpha(t))]\cdot N(\alpha(t)) = \dot{f}X + fX'$

8.2 ~~$\alpha(t)$ is always in S, i.e. $\vec{a}\cdot\alpha(t) = b$, i.e. $\dot{\alpha}(t) = 0$. $\dot{\alpha}(t) = 0 \in S^{\perp}_{\alpha(t)}$ So~~

Define Vector field (along α) $\vec{V}(t) = \vec{v}$ on S, As $v \in S_p$.

so V is tangent to S, $\dot{V} = 0$ (and of course along α). So V is parallel along α. So $P_\alpha(v) = (q, v)$

That means parallel transport in an n-plane is path independent

8.3 When $v_1 = (p, 1, 0, 0)$ By example on page 49, $V_1(t) = \dot{\alpha}(t) = (\cos t, 0, -\sin t)$, $V_1(\pi) = (-1, 0, 0)$

When $v_2 = (p, 0, 1, 0)$ Then the vector field $V_2(t) = (0,1,0)$, is parallel to S along α

So $V_2(\pi) = (0, 1, 0)$. As P_α is linear transform, $P_\alpha(v) = (q, -v_1, v_2, 0)$

8.4 Define geodesic $\alpha(t) = p\cos t + \hat{v}\sin t$, $\dot{\alpha}(t) = -p\sin t + \hat{v}\cos t$. $\alpha(0) = p$, $\alpha(\frac{\pi}{2}) = \hat{v}$

$\dot{\alpha}(0)\cdot v = \|v\|$, As $\dot{\alpha}(\frac{\pi}{2})\cdot P_\alpha(\hat{v}) = -p$ $P_\alpha(\hat{v}) = \|v\|$ and $\|P_\alpha(v)\| = \|v\|$, So $P_\alpha(v) = -p\|v\|$ (By corollary on Pg 48)

Likewise define geodesic $\beta(t) = p\sin t + \hat{w}\cos t$ $P_\beta(-p\|v\|) = w$ $\beta(\frac{\pi}{2}) = p$, $\beta(0) = \hat{w}$

So both $\alpha(\frac{\pi}{2})$ and $\beta(\frac{\pi}{2})$ are on $\{x \in S^2_p \mid p\cdot x = 0\}$ we define geodesic

(by example 3 in ch 7) Pg 40 $\nu(t) = \hat{v}\cos t + \sin t\cdot(p\times\hat{v})$, $\nu(0) = \hat{v}$, We now find t_0 s.t. $\nu(t_0) = \hat{w}$

$\hat{v}\cos t_0 + (p\times\hat{v})\sin t_0 = \hat{w} \Rightarrow \hat{v}\cdot\hat{w}\cdot\cos t_0 + p\cdot(\hat{v}\times\hat{w})\sin t_0 = 1$. Let the angle

between $\hat{v}$ and $\hat{w}$ be θ, since $p\perp\hat{v}$, $p\perp\hat{w}$, we have either

$\cos\theta\cos t_0 + \sin\theta\sin t_0 = 1$ or $\cos\theta\cos t_0 - \sin\theta\sin t_0 = 1$. But in whichever

case, there must be a solution t_0 ($t_0 = \theta$ or $-\theta$). Check $\nu(t) = -p\|v\|$ is parallel along $\nu(t)$:

$\dot{\nu}(t)\cdot p\|v\| = 0$ $\forall t$, $\|V(t)\| = \text{constant}$.

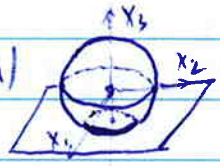
$V(t)\cdot N_{\nu(t)} = \pm p\|v\|\cdot(\hat{v}\cos t + (p\times\hat{v})\sin t) = 0$ So $V(t) \in S_{\nu(t)}$

Therefore $V(t) = -p\|v\|$ is parallel on S^2_p along $\nu(t)$ as $\nu(t)$ is geodesic (by Corollary Pg 48)

So we finally find a piecewise smooth pranetrized curve

$v \to P_\alpha(v) = -p\|v\| \to P_\nu(-p\|v\|) = -p\|v\| \to P_\beta(-p\|v\|) = w$.

8.5 (a)



$S_1 = \{(x - (0, 0, \frac{1}{2}))\cdot(0,0,1) = 0\}$.

$S_2 = \{x \mid \|x\|^2 = 1\}$. $\alpha(t) = (\frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{2}\sin t, \frac{1}{2})$

$X(t) = (\frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{2}\sin t, 0)$

$X(t)$ is parallel along α as viewed in S_1. But $X(t)$ is not parralel as viewed in S_2 Because $\dot{X}(t) = (-\frac{\sqrt{3}}{2}\sin t, \frac{\sqrt{3}}{2}\cos t, 0) \notin \{\lambda(0,0,1) \mid \lambda \in \mathbb{R}\} = S^{\perp}_{\alpha(t)}$

as $S_{1\alpha(t)}^{\perp} = S_{2\alpha(t)}^{\perp} \Leftarrow S_{1\alpha(t)} = S_{2\alpha(t)}$

(b) X is parallel along α in $S_1 \Leftrightarrow X(t) \in S_{1\alpha(t)}^{\perp} \Leftrightarrow X(t) \in S_{2\alpha(t)}^{\perp} \Leftrightarrow X$ is parallel in S_2

(c) α is geodesic in $S_1 \Leftrightarrow \ddot{\alpha}(t) \in S_{1\alpha(t)}^{\perp} \Leftrightarrow \ddot{\alpha}(t) \in S_{2\alpha(t)}^{\perp} \Leftrightarrow \alpha$ is geodesic in S_2

8.6 (a) Let $Y(u) = X(h(u))$

$\dot{X}(t) \in S_{\alpha(t)}^{\perp} \Rightarrow \dot{Y}(u) = \dot{X}(h(u))\, h'(u) \in S_{\alpha(h(u))}^{\perp} = S_{\beta(u)}^{\perp}$.

As $h'(t) \neq 0$. $h(t)$ is monotonic. so there is h^{-1}, $(h^{-1})' \neq 0$, $X = Y \circ h^{-1}$. So the same proof as above goes therefore it is iff.

(b) First by (a), $X \circ h$ is parallel along $\alpha \circ h$. If $\alpha(t_1) = p$, $\alpha(t_2) = q$.
Let $h(u_1) = t_1$, $h(u_2) = t_2$. So $\alpha(h(u_1)) = p$, $\alpha(h(u_2)) = q$.
$X(h(u_1)) = X(t_1)$, $X(h(u_2)) = X(t_2)$. Besides h is monotonic so $h: [u_1, u_2] \to [t_1, t_2]$
Thus, $X \cdot h$ transports p at u_1 to q at u_2

(c) $\forall v$. If $\alpha(t_1) = p$, $\alpha(t_2) = q$ X is parallel to S along α. $X(t_1) = v$. $X(t_2) = u$ (i.e. $P_\alpha(v) = u$)
$\beta(t) \triangleq \alpha(-t)$. As $\dot{X}(t) \in S_{\alpha(t)}^{\perp}$ ~~So $\dot{X}(-t) \in S_{\alpha(t)}^{\perp}$~~ Let $Y(t) = X(-t)$, then
$\dot{Y}(t) = -\dot{X}(-t) \in S_{\alpha(-t)}^{\perp} = S_{\beta(t)}^{\perp}$ ~~So $Y(t)$ is~~ Besides as $X(t) \cdot N_{\alpha(t)} = 0$
$Y(t) \cdot N_{\beta(t)} = X(-t) \cdot N_{\alpha(-t)} = 0$ So $Y(t)$ is parallel along $\beta(t)$
$\beta(-t_2) = q$. $\beta(-t_1) = q$. $Y(-t_2) = X(t_2) = u$. $Y(-t_1) = X(t_1) = v$
So u is transported to v along $\beta(t)$ from q at $-t_2$ to p at $-t_1$, i.e. parallel transport from q to p along $\alpha(-t)$ is the inverse of parallel transport from p to q along α

8.7 (i) γ is α concatenates with β. If P_α, P_β correspond to A and B respectively, then P_γ correspond to $A \cdot B$, which is also nonsingular

(ii) By the third question in Ex 8.6, P_α^{-1} is the parallel transport along $\alpha(-t)$, $\beta(t) \triangleq \alpha(-t)$ $t \in [-b, -a]$, P_β corresponds to A^{-1}

8.8 We use X^* to denote $X'(t)$, the Fermi derivative.

(a) i $(X+Y)^* = (X+Y)' - [((X+Y)'(t) \cdot \dot{\alpha}(t)]\dot{\alpha}(t)$ by $(X+Y)' = X' + Y'$
$= X' - [X'(t) \cdot \dot{\alpha}(t)]\dot{\alpha}(t) + Y' - [Y'(t) \cdot \dot{\alpha}(t)]\dot{\alpha}(t) = X^* + Y^*$

ii $(fX)^* = (fX)' - [(fX)' \cdot \dot{\alpha}(t)]\dot{\alpha}(t)$ (by $(fX)' = f'X + fX'$)
$= (f'X + fX') - [(f'X + fX') \cdot \dot{\alpha}(t)]\dot{\alpha}(t)$ (by $X(t) \cdot \dot{\alpha}(t) = 0$)
$= f'X + f\{X' - [X' \cdot \dot{\alpha}(t)]\dot{\alpha}(t)\} = f'X + fX^*$

iii $(X \cdot Y)' =$ ~~$(X \cdot Y)' - [(X \cdot Y)' \cdot \dot{\alpha}(t)]\dot{\alpha}(t)$ (by $(XY)' = X'Y + XY'$)~~
~~$= X'Y + XY' - [(X'Y + XY') \cdot \alpha(t)]\dot{\alpha}(t)$~~
$= X'Y + Y'X$

by $\dot{\alpha}(t) \cdot Y = \dot{\alpha}(t) X(t) = 0$

$X^*Y + X \cdot Y^* = [X' - (X'(t) \cdot \dot{\alpha}(t))\dot{\alpha}(t)]Y + X \cdot [Y' - (Y'(t) \cdot \dot{\alpha}(t))\dot{\alpha}(t)] = X'Y + XY'$

(b). By definition, we should have: $X\cdot\dot\alpha=0$. $X\cdot N\circ\alpha=0$ and $X^*=0$

$X^*=0 \Rightarrow \dot X-(\dot X\cdot N\circ\alpha)N\circ\alpha-(\dot X\cdot\dot\alpha)\dot\alpha+(\dot X\cdot N\circ\alpha)(N\circ\alpha\cdot\dot\alpha)\dot\alpha=0$ (*) Note: $\dot\alpha\perp N\circ\alpha$

$X\cdot\dot\alpha=0 \Rightarrow \dot X\dot\alpha+X\cdot\ddot\alpha=0$, $X\cdot N\circ\alpha=0 \Rightarrow \dot X\cdot N\circ\alpha+X\cdot N\dot\circ\alpha=0$

Plugging into (*): $\dot X+(X\cdot N\dot\circ\alpha)N\circ\alpha+(X\cdot\ddot\alpha)\dot\alpha=0$. 1st order differential equation together with initial condition $X(t_0)=V$. So there exists a unique solution $X(t)$.

Now check $X\cdot\dot\alpha=0$ and $X\cdot N\circ\alpha=0$

$\overset{\circ}{(X\cdot\dot\alpha)}=\dot X\dot\alpha+X\ddot\alpha=X\ddot\alpha-(X\cdot N\dot\circ\alpha)(N\circ\alpha\cdot\dot\alpha)-(X\cdot\ddot\alpha)(\dot\alpha\cdot\dot\alpha)=0$

$\overset{\circ}{(X\cdot N\circ\alpha)}=\dot X\cdot N\circ\alpha+X\cdot(N\dot\circ\alpha)=X\cdot(N\dot\circ\alpha)-(X\cdot N\dot\circ\alpha)(N\circ\alpha\cdot N\circ\alpha)-(X\cdot\ddot\alpha)(\dot\alpha\cdot N\circ\alpha)=0$

domain check ~~[illegible]~~ same as Thm 1 in chapter as $\|X\|$ is constant

(c) (i) F_α is linear map, If V and W are Fermi parallel along α, then so are V+W and cV $(c\in\mathbb{R})$.

(ii) F_α is one to one and onto: the kernel of F_α is zero because $\|F_\alpha(V)\|=0 \Leftrightarrow V=0$ by (iii).

so F_α is one-to-one from one n-dim vector space to another. But such maps are onto

(iii) $F_\alpha(V)\cdot F_\alpha(W)=V\cdot W$ $\forall V,W\in\dot\alpha(a)^\perp$ because $(X\cdot Y)^\bullet=X^*Y+XY^*=0$, i.e. $X\cdot Y$ is constant

9.1 (a) $\nabla f=(4x_1,6x_2)$ $\quad\nabla_v f(p)=(4,0)\cdot(2,1)=8$

(b) $\nabla f=(2x_1,-2x_2)$ $\quad\nabla_v f(p)=(2,-2)\cdot(\cos\theta,\sin\theta)=2(\cos\theta-\sin\theta)$

(c) $\nabla f=(x_2x_3^2,\ x_1x_3^2,\ 2x_1x_2x_3)$, $\nabla_v f(p)=(1,1,2)\cdot(a,b,c)=a+b+2c$

(d) $\nabla f=(q,2q)$ $\quad\nabla_v f(p)=$ ~~[illegible]~~ $2p\cdot v$

9.2 $\nabla_{e_i} f=\left(\frac{\partial f(p)}{\partial x_1},\cdots\frac{\partial f(p)}{\partial x_{n+1}}\right)\cdot(0,\cdots1,\cdots0)=\frac{\partial f(p)}{\partial x_i}$

9.3 (a) $\nabla X_1=(x_2,x_1)$ $\ \nabla X_2=(0,2x_2)$ $\quad\nabla_v X=((0,1)\cdot(0,1),(0,0)\cdot(0,1))=(1,0)$

(b) $\nabla X_1=(0,-1)$ $\ \nabla X_2=(1,0)$. $\nabla_v X=((0,-1)(-\sin\theta,\cos\theta),(1,0)(-\sin\theta,\cos\theta))=(-\cos\theta,-\sin\theta)$

(c) $\nabla X_i=(q,2e_i)$ $\quad\nabla_v X=(2,2,\cdots2)$.

9.5 Let $Y(t)=X(\alpha(t))$ be the vector field tangent to S along α. As $D_vX=(X\circ\alpha)'(t_0)$ [$F(t)=f(\alpha(t))$, $D_vf=F'(t_0)$]

where $\alpha: I\to S$ is any parametrized curve in S with $\dot\alpha(t_0)=v$. Then quote the properties i-iii in chapter 8 on page 46. Note in (iii) $\nabla_v(X\cdot Y)$ rather than $D_v(X\cdot Y)$ ($\nabla_v X\cdot Y=(X\cdot Y)'$)

9.4 Same as 9.5. $\nabla_v X=(X\circ\alpha)(t_0)$ $\quad\nabla_v f=\dot F$. Then quote the properties i-iii in chapter 7 on Pg 39

9.6. $X(q)\cdot X(q)=1$ By property iii of ch 9 on Pg 54. $\nabla_v X(q)\cdot X(q)=\nabla_v 1=0$ i.e. $\nabla_v X\perp X(p)$

If X is tangent to S, then $\nabla_v X \cdot N(p) = (\dot{X\circ\alpha})(t_0)\cdot N(p) = 0$. So $\nabla_v X = D_v X$. so $D_v X \perp X(p)$

by proof in Thm 1 of chapter 5, then α is ~~unique~~ on S

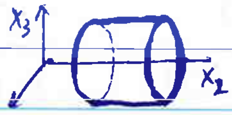
9.7 (a) "if part": $\forall$ ~~parametric~~ integral curve $\alpha: I \to S$, s.t. $\alpha(t_0) = p$, $\dot\alpha(t) = X(\alpha(t))$, ~~then the geodesic is~~

$D_{X(p)}X = 0 \Leftrightarrow \nabla_{X(p)}X \,//\, N(p) \Leftrightarrow (\dot{X\circ\alpha}) \,//\, N\circ\alpha$
$\dot\alpha(t) = X\circ\alpha \Rightarrow \ddot\alpha = (\dot{X\circ\alpha})$
$\Rightarrow \ddot\alpha \,//\, N\circ\alpha \Rightarrow \alpha$ is geodesic.

"only if" part: $\forall p \in S$, construct integral curve $\alpha: I \to$, s.t. $\alpha(t_0) = p$, $\dot\alpha(t) = X(\alpha(t))$

as X is tangent to S, α must be on S by proof in Thm 1 in chapter 5. By assumption $\ddot\alpha \in S_\alpha^\perp$ (geodesic)

As $\dot\alpha(t) = X(\alpha(t))$, we have $\ddot\alpha = (\dot{X\circ\alpha}) \in S_\alpha^\perp$ i.e. $(\dot{X\circ\alpha}) \,//\, N\circ\alpha \Leftrightarrow D_{X(p)}X = 0$

(b) 

cylinder $X(p) = (p, (0,1,0))$

9.8 (a) $N = (a_1, \dots, a_{n+1})$, $\nabla N_i = 0$, $L_p(v) = 0$

(b) $N = (0, \frac{x_2}{a}, \frac{x_3}{a})$. $\nabla N_1 = (0,0,0)$, $\nabla N_2 = (0, \frac{1}{a}, 0)$, $\nabla N_3 = (0,0,\frac{1}{a})$, $L_p(v) = -(0, \frac{v_2}{a}, \frac{v_3}{a})$ (let $a > 0$)

9.9 By property (ii) on page 54. $\nabla_v(-N) = \nabla_v(-1)\cdot(N) + (-1)\nabla_v(N) = -\nabla_v N$

9.10 (a) suppose $L^*(e_i) = \sum_{i=1}^n \lambda_i e_i$, then by $L^*(e_i)\cdot e_j = e_j \cdot L(e_i)$ we have $\lambda_j = e_j \cdot L(e_i)$

So $L^*(e_i) = \sum_{j=1}^n (e_i \cdot L(e_j))\, e_j$, $\forall v = \sum_{i=1}^n \alpha_i e_i \in V$, $L^*(v) = \sum_{i=1}^n \alpha_i \sum_{j=1}^n (e_i \cdot L(e_j))\cdot e_j$.

$L^*(v)\cdot w = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \alpha_i \beta_k (e_i \cdot L(e_j))\cdot(e_j \cdot e_k) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j (e_i \cdot L(e_j))$ Let $w = \sum_{i=1}^n \beta_i e_i$

$v \cdot L(w) = \sum_{i,j=1}^n \beta_i \alpha_j\, L(e_i)\cdot e_j = \sum \alpha_i \beta_j (e_i \cdot L(e_j)) = L^*(v)\cdot w$

So the only possible choice of $L^*(e_i) = \sum_{j=1}^n (e_i \cdot L(e_j))\, e_j$ satisfies $v \cdot L(w) = L^*(v)\cdot w$ $\forall v, w \in V$.

(b) if ~~$L^* = L$~~ $L(v) = Av$ $\forall v, w \in V$. For $\forall w \in V$, $w \cdot L(v) = wAv$,

If we ~~set~~ choose $L^*(v) = A^T v$. then $v \cdot L^*(w) = v \cdot A^T w = wAv = wL(v)$.

As (a) proves L^* is unique and each linear transform corresponds to a unique matrix we know L^* correspond to A'. So $L^* = L \Leftrightarrow A$ is symmetric. So L_p is symmetric by Thm 2 (Pg 58)

9.11 $\forall i \in \{1, \dots, n\}$, $L_p(e_i) = -\nabla_{e_i} N(p) = -((\nabla N_1(p)\cdot e_i), \dots, (\nabla N_{n+1}(p)\cdot e_i))$

$\forall j \in \{1, \dots, n\}$, $\nabla N_j(p)\, e_i = \frac{\partial N_j}{\partial x_i}\Big|_p = \frac{\partial}{\partial x_i}\left(\frac{1}{\|\nabla f\|}\frac{\partial f}{\partial x_j}\right)\Big|_p = \frac{\partial}{\partial x_i}\left(\frac{1}{\|\nabla f\|}\right)\Big|_p \cdot \frac{\partial f}{\partial x_j}\Big|_p + \frac{1}{\|\nabla f\|}\frac{\partial^2 f}{\partial x_i \partial x_j}$

Since S is n-surface $\left|\frac{\partial}{\partial x_i}\left(\frac{1}{\|\nabla f\|}\right)\right|_p < \infty$. But $\nabla f(p)/\|\nabla f(p)\| = e_{n+1} \Rightarrow \frac{\partial f}{\partial x_j}\Big|_p = 0$ $\forall j \in \{1, \dots, n\}$

So $\nabla N_j(p)\, e_i = \frac{1}{\|\nabla f\|}\frac{\partial^2 f}{\partial x_i \partial x_j}$

Since $N(p) = e_{n+1}$, and L_p is map $S_p \mapsto S_p$. So $\nabla N_{n+1}(p)\cdot e_i = 0$, thus

$L_p(e_i) = -\sum_{j=1}^n \frac{1}{\|\nabla f\|}\frac{\partial^2 f}{\partial x_i \partial x_j}\, e_j$

By the way, we can prove that $\nabla N_{n+1}(p)\cdot e_i = 0$. First $\frac{\partial f}{\partial x_{n+1}}\Big|_p = \|\nabla f(p)\|$

Second. $\frac{\partial}{\partial x_i}\|\nabla f\|\Big|_p = \frac{\partial}{\partial x_i}\left[\sum_{k=1}^{n+1}\left(\frac{\partial f}{\partial x_k}\right)^2\right]^{\frac{1}{2}}\Big|_p = \left[\sum_{k=1}^{n+1}\left(\frac{\partial f}{\partial x_k}\right)^2\right]^{-\frac{3}{2}}\cdot\frac{-1}{2}\cdot\sum_{k=1}^{n+1} 2\frac{\partial f}{\partial x_k}\frac{\partial^2 f}{\partial x_k \partial x_i}\Big|_p$. But $\frac{\partial f}{\partial x_k}\Big|_p = \begin{cases} 0 & k=1,\cdots,n \\ \|\nabla f\| & k=n+1\end{cases}$

$= -\|\nabla f\|^{-3}\cdot\|\nabla f\|\frac{\partial^2 f}{\partial x_{n+1}\partial x_i} = -\|\nabla f\|^{-2}\frac{\partial^2 f}{\partial x_{n+1}\partial x_i}$ So L_p is symmetric

So $\nabla N_{n+1}(p)\cdot e_i = -\|\nabla f\|^{-2}\frac{\partial^2 f}{\partial x_{n+1}\partial x_i}\|\nabla f\| + \|\nabla f\|^{-1}\frac{\partial^2 f}{\partial x_{n+1}\partial x_i} = 0$.

9.12 (a) Suppose a parametrized curve $\alpha: I\to S$. $\alpha(t_0)=p$. $\dot\alpha(t_0)=X(\alpha(t_0))$

$\nabla_{X(p)}Y = \dot{(Y\circ\alpha)}$, $\nabla_{X(p)}Y\cdot N(p) = \dot{(Y\circ\alpha)}\cdot N\circ\alpha$.

But as Y is tangent to S. $(Y\circ\alpha)\cdot(N\circ\alpha)=0 \Rightarrow (\dot{Y\circ\alpha})(N\circ\alpha) + (Y\circ\alpha)(\dot{N\circ\alpha}) = 0$

So $\nabla_{X(p)}Y\cdot N(p) = -(Y\circ\alpha)\cdot(\dot{N\circ\alpha})\Big|_{t_0} = -Y(p)\cdot L_p(X(p))$

Similarly, one can prove $\nabla_{Y(p)}X\cdot N(p) = -X(p)\cdot L_p(Y(p))$

By Thm 2 (Pg 55) $L_p(X(p))\cdot Y(p) = X(p)\cdot L_p(Y(p))$ Thus $\nabla_{X(p)}Y\cdot N(p) = \nabla_{Y(p)}X\cdot N(p)$

(b) by (a) obvious

9.13. For $\forall v$, define a parametrized curve $\alpha: I\to U$, $\alpha(t_0)=p$. $\dot\alpha(t_0)=v$. For $\forall\varepsilon$ there exists a δ s.t. $\|X(p+v)-X(p)-X'(p)(v)\|/\|v\|<\varepsilon$, $\forall \|v\|<\delta$. As α is continuous, there exists $\delta_1>0$ s.t. $\forall t\in(t_0-\delta_1, t_0+\delta_1)$: $\|\alpha(t)-\alpha(t_0)\|<\delta$. thus

$\|X(p+\alpha(t)-\alpha(t_0)) - X(p) - X'(p)(\alpha(t)-\alpha(t_0))\| / \|\alpha(t)-\alpha(t_0)\| < \varepsilon$

i.e. $\|X(\alpha(t)) - X(\alpha(t_0)) - X'(p)(\alpha(t)-\alpha(t_0))\| / \|\alpha(t)-\alpha(t_0)\| < \varepsilon$.

i.e. $\lim_{t\to t_0}\|X(\alpha(t)) - X(\alpha(t_0)) - X'(p)(\alpha(t)-\alpha(t_0))\| / \|\alpha(t)-\alpha(t_0)\| = 0$ (*)

Notice $\lim_{t\to t_0}\frac{\alpha(t)-\alpha(t_0)}{t-t_0} = \dot\alpha(t) = v$ So $\lim_{t\to t_0}\frac{\|\alpha(t)-\alpha(t_0)\|}{|t-t_0|} = \|v\|$ (1)

$\lim_{t\to t_0}(X(\alpha(t)) - X(\alpha(t_0)))/(t-t_0) = \nabla_v X$ (by definition of $\nabla_v X$) (2)

As $X'(p)$ is a linear map, ~~suppose its corresponding matrix is A~~, thus

If $\lim_{i\to\infty} V_i = V$ then $\lim_{i\to\infty} A(V_i) = A(V)$ use basis expression must finite dimensional

$\lim_{t\to t_0}\frac{X'(p)(\alpha(t)-\alpha(t_0))}{t-t_0} = \lim_{t\to t_0} X'(p)\left(\frac{\alpha(t)-\alpha(t_0)}{t-t_0}\right)$ (since $X'(p)$ is linear)

$= X'(p)\left(\lim_{t\to t_0}\frac{\alpha(t)-\alpha(t_0)}{t-t_0}\right) = X'(p)(v)$ (3)

Plugging (1) (2) (3) into (*) $\|\nabla_v X - X'(p)(v)\|/\|v\| = 0$ i.e. $\nabla_v X = X'(p)(v)$

9.14 $L_p(p,v) \overset{\text{def of } L_p}{=} -\nabla_v N(p) \overset{\text{def. of } \tilde N}{=} -\nabla_v \tilde N(p) \overset{\text{result of 9.13}}{=} -\tilde N'(p)(v)$

9.15 (a) $\dot{\tilde\alpha}(t) = X(\tilde\alpha(t)) \Leftrightarrow \begin{cases} \dot x_1 = u_1, \cdots, \dot x_{n+1} = u_{n+1} \\ \dot u_k = -(u_1,\cdots,u_{n+1})\cdot\left(\sum_{i=1}^{n+1} u_i\frac{\partial N_1}{\partial x_i}, \cdots, \sum_{i=1}^{n+1} u_i\frac{\partial N_{n+1}}{\partial x_i}\right)N_k = -N_k\sum_{i,j=1}^{n+1} u_i u_j\frac{\partial N_j}{\partial u_i}\end{cases}$ (denote $\alpha=(x_1,\cdots,x_{n+1})$, $\tilde\alpha=(\alpha, u_1,\cdots,u_{n+1})$)

So $\ddot x_k + N_k\cdot\sum_{i,j=1}^{n+1}\dot x_i\dot x_j\frac{\partial N_j}{\partial x_i} = 0$ which is the same as (G) f

Then follow the proof in the theorem of chapter 7, α is a geodesic of S. (Note: $\alpha\subset S$ is assumed)

Note the equation $\dot{\tilde\alpha}(t) = X(\tilde\alpha(t))$ is 1st order differential system in u and x so unique solution

(b) $X(\beta(t)) = \dot\beta(t) \Leftrightarrow \left(\dot\beta_1 = \beta_2 \text{ and } \dot\beta_2 = -(\beta_2\cdot\nabla_{\beta_2}N)N(\beta_1)\right.$ As in (a), we can derive the equation (G) in terms of β_1. (G) itself guarantees β_1 is on S as shown by the proof in Thm of chapter 7, given that $\beta_1(t_0) = p \in S$, $\dot\beta_1(t_0) = \beta_2(t_0) = v \in S_p$.

10.1 $\alpha = (x, y)$ $\dot\alpha = (x', y')$, $\ddot\alpha = (x'', y'')$ $N = (-y', x')$ (due to consistency).

So $k \circ \alpha = \ddot\alpha \cdot N\circ\alpha / \|\dot\alpha\|^2 = (-x''y' + y''x') / (x'^2 + y'^2)^{3/2}$

10.2 $f: x_2 - g(x_1)$, ~~$f' = (-g'(x_1), 1)$~~ $f^{-1}(0)$ can be viewed as $\alpha(t) = \begin{cases} t \\ g(t) \end{cases}$ $t \in I$

By Ex 10.1. curvature of C at point $(t, g(t)) = k \circ \alpha = g''(t) / [1 + (g'(t))^2]^{3/2}$

10.3 (a) $\nabla = (a, b)$ $X = (+b, -a)$ $\dot\alpha(t) = X(\alpha(t)) \Rightarrow \alpha(t) = \begin{pmatrix} +bt + c_1 \\ -at + c_2 \end{pmatrix}$, ~~let $\alpha(0) = \frac{c}{2}\begin{pmatrix} 1/a \\ 1/b \end{pmatrix} \Rightarrow \alpha(t) = \begin{pmatrix} -bt + \frac{c}{2a} \\ at + \frac{c}{2b} \end{pmatrix}$~~

Since $(a, b) \neq (0, 0)$ let $a \neq 0$. let $\alpha(0) = \begin{pmatrix} c/a \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 = c/a \\ c_2 = 0 \end{pmatrix} \Rightarrow \alpha(t) = \begin{pmatrix} +bt + c/a \\ -at \end{pmatrix}$ $t \in R$

(b) $\nabla = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}\right)$ $X = \left(\frac{2x_2}{b^2}, \frac{-2x_1}{a^2}\right)$ $\dot\alpha(t) = X(\alpha(t))$, $\frac{1}{a^2}\alpha_1^2(t) + \frac{1}{b^2}\alpha_2^2(t) = 1$ $\Rightarrow \begin{cases} \alpha_1(t) = a \sin \frac{2}{ab} t \\ \alpha_2(t) = b \cos \frac{2}{ab} t \end{cases}$ $t \in R$

(c) $\nabla = (-2ax_1, 1)$, $X = (1, 2ax_1)$, $\dot\alpha(t) = X(\alpha(t)) \Rightarrow \begin{cases} \alpha_1(t) = t + c_1 \\ \alpha_2(t) = at^2 + 2ac_1t + c_2 \end{cases}$

$\alpha_2(t) - a(\alpha_1(t))^2 = c \Rightarrow c_2 = c + a c_1^2$. let $c_1 = 0$, $c_2 = c$. So $\begin{cases} \alpha_1(t) = t \\ \alpha_2(t) = at^2 + c \end{cases}$ $t \in R$

(d) $\nabla = (+2x_1, -2x_2)$ $X = (-2x_2, -2x_1)$ $\dot\alpha(t) = X(\alpha(t))$, $\alpha_1^2 - \alpha_2^2 = 1$ $\Rightarrow$ ~~$\alpha_1(t) = \cos t$, $\alpha_2(t) = \sin t$~~ ~~$t \in [0, 2\pi) \setminus \{\frac{\pi}{2}, \frac{3\pi}{2}\}$~~

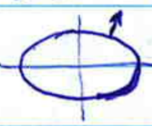
$\alpha_1(t) = \frac{c_1e^{2t} + c_2e^{-2t}}{2}$ ~~$\alpha_1(t) = \frac{1}{2}(e^{2t} + e^{-2t})$, $\alpha_2(t) = \frac{c_2}{2}(e^{-2t} - e^{2t})$~~

$\alpha_2(t) = \frac{c_2e^{-2t} - c_1e^{2t}}{2}$ $\forall$ ~~$c_1, c_2 \in \mathbb{R}$~~ if ignore $x_1 > 0$ $c_1c_2 = 1$

10.4 (a) $k = 0$ as $\ddot\alpha = 0$. (b) $\alpha = \begin{pmatrix} a \sin 2t/ab \\ b \cos 2t/ab \end{pmatrix}$, $\dot\alpha = \begin{pmatrix} 2/b \cos(2t/ab) \\ -2/a \sin(2t/ab) \end{pmatrix}$, $\ddot\alpha = \begin{pmatrix} -4/ab^2 \sin(2t/ab) \\ -4/a^2b \cos(2t/ab) \end{pmatrix}$

$N = \lambda \begin{pmatrix} 2/a \sin(2t/ab) \\ 2/b \cos(2t/ab) \end{pmatrix}$ ~~$= \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} b \sin(2t/ab) \\ a \cos(2t/ab) \end{pmatrix}$~~, $k(p) = \frac{\ddot\alpha \cdot N\circ\alpha}{\|\dot\alpha\|^2}$ ~~$= \frac{-4/ab}{4(a^2+b^2)/a^2b^2} = \frac{-ab}{a^2+b^2}$ $\ddot\alpha \cdot N\circ\alpha = \frac{-4}{ab\sqrt{a^2+b^2}}$~~

$\|\dot\alpha\|^2 = \frac{4}{a^2b^2}\left(a^2\cos^2\frac{2t}{ab} + b^2\sin^2\frac{2t}{ab}\right)$



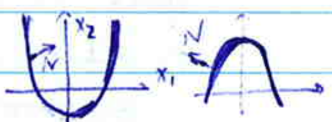
So $k < 0$

$\ddot\alpha \cdot N\circ\alpha = \frac{-4}{a^2b^2}\begin{pmatrix} a \sin\frac{2t}{ab} \\ b \cos\frac{2t}{ab} \end{pmatrix} \cdot \frac{2}{ab}\begin{pmatrix} b \sin\frac{2t}{ab} \\ a \cos\frac{2t}{ab} \end{pmatrix} \Big/ \frac{2}{ab}\sqrt{a^2\cos^2\frac{2t}{ab} + b^2\sin^2\frac{2t}{ab}}$

So $k(p) = \frac{\ddot\alpha \cdot N\circ\alpha}{\|\dot\alpha\|^2} = -ab\left(a^2\cos^2\frac{2t}{ab} + b^2\sin^2\frac{2t}{ab}\right)^{-3/2}$ If $a = b = r$, then $k(p) = -\frac{1}{r}$.

$= -ab\left(\frac{a^2}{b^2}x_2^2 + \frac{b^2}{a^2}x_1^2\right)^{-3/2}$

(c) Use Ex 10.2. $g(t) = at^2 + c$, $g'(t) = 2at$, $g''(t) = 2a$

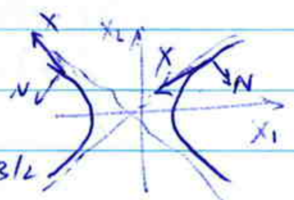


$k \circ \alpha = 2a / (1 + 4a^2t^2)^{3/2} = 2a / (1 + 4a^2x_1^2)^{3/2}$

(d) Use Ex 10.1 ~~$\alpha(t) = \begin{pmatrix} \frac{1}{\cos t} \\ \operatorname{tg} t \end{pmatrix}$ $\dot\alpha(t) = \begin{pmatrix} \sin t / \cos^2 t \\ 1/\cos^2 t \end{pmatrix}$~~ $\ddot\alpha(t) = \frac{1}{\cos^3 t}\begin{pmatrix} 1 + \sin^2 t \\ 2\sin t \end{pmatrix}$

~~$k \circ \alpha = -\cos^3 t / (1 + \sin^2 t)^{3/2} = -(x_1^2 + x_2^2)^{3/2} \cdot \operatorname{sgn}(x_1)$~~

In general for $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $k = -ab / (a^2\operatorname{tg}^2 t + b^2\sec^2 t)^{3/2}$



$\alpha(t) = \frac{1}{2}(c_1e^{2t} + c_2e^{-2t}, c_2e^{-2t} - c_1e^{2t})^T$, $\dot\alpha(t) = (c_1e^{2t} - c_2e^{-2t}, -c_2e^{-2t} - c_1e^{2t})^T$

$\ddot\alpha(t) = 2(c_1e^{2t} + c_2e^{-2t}, c_2e^{-2t} - c_1e^{2t})$ So $k \circ \alpha = 8 / [2(e^{4t} + e^{-4t})]^{3/2}$

$k = 1 / (x_1^2 + x_2^2)^{3/2}$, So curve is always turning (according to X) towards N

10.5 $h(t_0) = (\alpha(t_0) - p)\cdot N(p) = (p-p)\cdot N(p) = 0$ $h'(t_0) = (\dot\alpha(t_0))\cdot N_p = 0$.

$h''(t_0) = \ddot\alpha(t_0)\cdot N(p) = k(p)$ because $\|\dot\alpha(t_0)\| = 1$

10.6 (a) As $\|\dot\alpha\| = \text{const}$ $\dot\alpha\cdot\ddot\alpha = 0$ But $\dot\alpha\cdot N\circ\alpha = 0$ and $\{v \mid v\cdot\dot\alpha = 0\}$ is one dimensional (as C is in 2D plane) So $\ddot\alpha = \lambda N\circ\alpha$, $\lambda = \ddot\alpha\cdot N\circ\alpha = k\circ\alpha$, So $\dot T = \ddot\alpha = (k\circ\alpha)\cdot(N\circ\alpha)$

(b) $\|N\| = 1$. So $(N\circ\alpha)\cdot(\dot N\circ\alpha) = 0$. But $(N\circ\alpha)\cdot\dot\alpha = 0$. and we are in 2-D plane

so $\dot N\circ\alpha = \lambda\dot\alpha$ $\lambda = \dot N\circ\alpha\cdot\dot\alpha$ Besides, as $\dot\alpha\cdot N\circ\alpha = 0$ we have

$\ddot\alpha\cdot N\circ\alpha + \dot\alpha\cdot\dot N\circ\alpha = 0$ So $\lambda = -\ddot\alpha\cdot N\circ\alpha = -k\circ\alpha$.

Thus. $\dot N\circ\alpha = -(k\circ\alpha)\cdot\dot\alpha = -(k\circ\alpha)\cdot T$.

10.7 (a) $\|\dot\alpha\| = 1 \Rightarrow \dot\alpha\ddot\alpha = 0 \Rightarrow T\perp N$. $B\perp N$ and $B\perp T$ are by definition of B (cross product)

(b) $\dot T = \ddot\alpha \,/\!/\, N(t) = \ddot\alpha/\|\ddot\alpha\|$ So $\dot T = \|\ddot\alpha\|\cdot N$ so $k \triangleq \|\ddot\alpha\|$

$\dot B = \dot T\times N + T\times\dot N = T\times\dot N$ So $\dot B\perp T$. $\dot B\perp\dot N$ But we know $N\perp T$

and $\|N\| = 1 \Rightarrow \dot N\perp N$. As we are in 3-D space $\dot B = -\tau\cdot N$ wher τ $I\to R$

$\tau(t) = -\dot B(t)\cdot N(t)$ so $\tau(t)$ is smooth.

$\dot N\perp N$. We know $B\perp N$, $T\perp N$ and $B\perp T$. So there exist $\lambda_1,\lambda_2 : I\to R$

$\dot N = \lambda_1 B + \lambda_2 T$ $\lambda_1 = \dot N\cdot B = -N\cdot\dot B = \tau$ (Since $N\cdot B = 0 \Rightarrow \dot NB + N\dot B = 0$)

$\lambda_2 = \dot N\cdot T = -N\cdot\dot T = -k$ (since $N\cdot T = 0 \Rightarrow \dot N\cdot T + N\dot T = 0$)

So $\dot N = \tau B - kT$.

10.8 By definition of circle of curvature. $C_p = O_p$. $\dot\alpha(0)\in C_p$ $\dot\beta(0)\in O_p$ C_p and O_p are one dimensional, $\|\dot\alpha(0)\| = \|\dot\beta(0)\| = 1$ So $\dot\alpha(0) = \pm\dot\beta(0)$. Furthermore $\dot\alpha(0), \dot\beta(0)$ are both consistent with $N(p)$ and $N_1(p)$ resp. ($N(p)$ and $N_1(p)$ are orientation norms of C and O). But $N_1(p) = N(p)$

Thus $\dot\alpha(0) = \dot\beta(0)$

As ~~$\ddot\alpha\perp\dot\alpha \Rightarrow \ddot\alpha \,/\!/\, N(p)$~~ $\ddot\alpha\cdot N(p) = -\nabla_{\dot\alpha(0)}N\cdot\dot\alpha(0)$ by Thm 1 of chapter 9

~~$\ddot\beta\perp\dot\beta \Rightarrow \ddot\beta \,/\!/\, N(p)$~~ $\ddot\beta\cdot N_1(p) = -\nabla_{\dot\beta(0)}N_1\cdot\dot\beta(0)$

But $\dot\alpha(0) = \dot\beta(0)$ and by definition of circle of curvature, $\nabla_{\dot\alpha(0)}N = \nabla_{\dot\beta(0)}N_1$

So $\ddot\alpha(0)N(p) = \ddot\beta(0)\cdot N_1(p)$ (*) ~~But $N = N_1(p)$~~ As $\ddot\alpha \,/\!/\, N(p)$, suppose

$\ddot\alpha(0) = \lambda_1 N(p)$, suppose $\ddot\beta(0) = \lambda_2 N_1(p)$ similarly as $\ddot\beta \,/\!/\, N_1(p)$

So $\lambda_1 = \ddot\alpha(0)\cdot N(p) \overset{\text{by }(*)}{=} \ddot\beta(0)\cdot N_1(p) = \lambda_2$, $\ddot\alpha(0) = \lambda_1 N(p) = \lambda_2 N_1(p) = \ddot\beta(0)$ (as $N_1(p) = N(p)$)

10.9 "only if": $O: \|x - q\|^2 = r^2$. $p\in O \Rightarrow \|p-q\|^2 = r^2 \Rightarrow f(0) = \|p-q\|^2 - r^2 = 0$

$C_p = O_p \Rightarrow$ the normal vector of O at $p = 2(p-q)\perp O_p = C_p = \{\lambda\dot\alpha(0) \mid \lambda\in R\}$

so $(p-q)\cdot\dot\alpha(0)=0$ so $f'(0)=2(\alpha(0)-q)\dot\alpha(0)=2(p-q)\cdot\dot\alpha(0)=0$.

By Thm 1 of Chapter 9. $\ddot\alpha(t_0)\cdot N(p) = -\nabla_{\dot\alpha(0)} N(p)\cdot\dot\alpha(0)$ (N, N_1 are orientation of C and O resp.)

$N(p)=N_1(p)=\lambda(p-q)/r$ ($\lambda=\pm1$ which determines orientation) $\lambda=1$ outwards $\lambda=-1$ inwards

$\nabla_v N(p)=\nabla_v N_1(p)=\lambda\frac{1}{r}v$.

So $\ddot\alpha(t_0)\lambda(p-q)/r=\ddot\alpha(t_0)\cdot N(p) = -\nabla_{\dot\alpha(0)}N(p)\cdot\dot\alpha(0) = -\nabla_{\dot\alpha(0)}N_1(p)\cdot\dot\alpha(0) = -\lambda\frac{1}{r}\dot\alpha(0)\cdot\dot\alpha(0)=\frac{-\lambda}{r}$

So $\ddot\alpha(p-q)=-1$. So $f''(0)=2+2(p-q)\ddot\alpha(0)=0$

"If part" $f(0)=0 \Rightarrow \|p-q\|=r^2$ So $p\in O$. ①

$f'(0)=0\Rightarrow(p-q)\cdot\dot\alpha(0)=0$ As we are in 2D, and $p-q\in O_p^\perp$. So $\dot\alpha(0)\in O_p$.

But $\dot\alpha(0)\in C_p$ as well and O_p and O_p are both one dimensional, So $O_p=C_p$ ②, then we can easily choose an orientation of O such that its orientation at p is the same as C's ③

$f''(0)\Rightarrow(p-q)\cdot\ddot\alpha(0)=-1$ ~~$\forall v\in C_p$, i.e. $v=\mu\dot\alpha(0)$, N~~

Since $\nabla_v N\cdot N=0$ and $N\perp\dot\alpha(0)$ So $\nabla_{\dot\alpha(0)}N=a\cdot\dot\alpha(0)$ $a\in\mathbb{R}$ as we are in 2D

$a=\nabla_{\dot\alpha(0)}N\cdot\dot\alpha(0)=-\ddot\alpha(0)N(p)=-\ddot\alpha(0)N_1(p)=-\lambda(p-q)/r\,\ddot\alpha(0)=\frac{\lambda}{r}$,

So $\nabla_{\dot\alpha(0)}N=\frac{\lambda}{r}\dot\alpha(0)$. But $\nabla_{\dot\alpha(0)}N_1=\frac{\lambda}{r}\dot\alpha(0)$ By Example in chapter 9 or page 56

So $\nabla_{\dot\alpha(0)}N=\nabla_{\dot\alpha(0)}N_1$. Furthermore, $\forall v\in C_p$, v must be $v=\mu\dot\alpha(0)$ $\mu\in R$.

But $\nabla_v N=\nabla_{\mu\dot\alpha(0)}N=\mu\cdot\nabla_{\dot\alpha(0)}N=\mu\nabla_{\dot\alpha(0)}N_1=\nabla_{\mu\dot\alpha(0)}N_1=\nabla_v N_1$ ④

Combining ①-④. O is circle of curvature of C at p.

10.10 $\dot\alpha(t)=(\cos\theta(t),\sin\theta(t))$ As $\alpha(t)$ is local parametrization of C

$N(\alpha(t))=(-\sin\theta(t),\cos\theta(t))$. $\ddot\alpha(t)=(-\sin\theta(t)\cdot\dot\theta(t),\cos\theta(t)\dot\theta(t))$. As α is unit speed, $k\circ\alpha=\ddot\alpha(t)\cdot N(\alpha(t))=\dot\theta(t)\triangleq d\theta/dt$.

11.1 $l(\alpha)=\int_0^2\|(2t,3t^2)\|dt=\int_0^2\sqrt{4+9t^2}\,t\,dt \overset{u=t^2}{=} \frac{1}{2}\int_0^4\sqrt{4+9u}\,du$
$=\frac{1}{18}\int_0^4\sqrt{4+9u}\,d(4+9u)=\frac{1}{18}\frac{2}{3}(4+9u)^{\frac{3}{2}}\Big|_0^4=\frac{10}{27}(10\sqrt{10}-1)$

11.2 $l(\alpha)=\int_{-1}^1\|(-3\sin3t,3\cos3t,4)\|dt=10$

11.3 $l(\alpha)=\int_0^{2\pi}\|-2\sqrt2\sin2t,2\cos2t,2\cos2t\|dt=\int_0^{2\pi}2\sqrt2\,dt=4\pi\sqrt2$

11.4 $l(\alpha)=\int_0^{2\pi}\|(-\sin t,\cos t,-\sin t,\cos t)\|dt=2\sqrt2\pi$

11.5. $\alpha(t)=(12t,-5t)$ $t\in(-1,1)$ $l(C)=l(\alpha)=\int_{-1}^1 13\,dt=26$ Ref Ex 11.9 changes sign

Actually, don't bother with orientation and $\dot\alpha$ compliance, because $l(C)\geq0$.

11.6 $\alpha(t) = (2\sin t, 1+2\cos t)$ $l(C) = l(\alpha) = \int_0^{2\pi} \|\dot\alpha(t)\| dt = \int_0^{2\pi} \|(2\cos t, -2\sin t)\| dt = 4\pi$

11.7 $\alpha(t) = (\sqrt{1+t^2}, t)$, $t \in (-\sqrt3, \sqrt3)$, $l(C) = l(\alpha) = \int_{-\sqrt3}^{\sqrt3} \|(t(1+t^2)^{-1/2}, 1)\| dt$
$= 2\int_0^{\sqrt3} \sqrt{1+t^2/(1+t^2)}\, dt$

11.8 $\alpha(t) = (\frac{2}{3}t^{\frac32}, t)$ $t \in (0,3)$ $l(C) = l(\alpha) = \int_0^3 \|(t^{\frac12}, 1)\| dt = \int_0^3 \sqrt{t+1}\, d(t+1) = 14/3$

11.9 If $\alpha(t)$ $t \in (a,b)$ is consistent with N, then $\beta(t) \triangleq \alpha(-t)$ $t \in (-b,-a)$ is consistent with $-N$
$(\dot\alpha_1(t), \dot\alpha_2(t))^T = R_{-\pi/2}(N_1(\alpha(t)), N_2(\alpha(t)))^T$. for $\forall t \in (a,b)$
$(\dot\beta_1(t), \dot\beta_2(t))^T = (-\dot\alpha_1(-t), -\dot\alpha_2(-t))^T = R_{-\pi/2}(-N_1(\alpha(-t)), -N_2(\alpha(-t)))^T$
$\int_{-b}^{-a} \alpha(-t) dt = \int_a^b \alpha(t) dt$ So $l(C) = l(\tilde C)$

11.10 (a) $\int_a^b |k \circ \alpha(t)| dt = \int_a^b |\ddot\alpha \cdot N(\alpha(t))| dt = \int_a^b \|\ddot\alpha(t)\| dt$. If β is reparametrization of α. $\beta: (c,d) \to C$
$\beta = \alpha \circ h$. (Since α, β are both one-to-one, such h must exist, $h(t) \triangleq \alpha^{-1}(\beta(t))$) since both α and β are ~~smooth~~ regular. ($\dot\alpha \neq 0$, $\dot\beta \neq 0$) h must be differentiable
$\dot\beta(t) = \dot\alpha(h(t)) \cdot h'(t)$. But $\|\dot\alpha\| = \|\dot\beta\| = 1$, so $\|h'(t)\| \equiv 1$. But h' is continuous.
So $h' \equiv 1$ or $h' \equiv -1$. In whichever case $\ddot\beta(t) = \ddot\alpha(h(t))(h'(t))^2 = \ddot\alpha(h(t))$
so $\int_c^d |k \circ \beta(t)| dt = \int_c^d \|\ddot\beta(t)\| dt = \begin{cases} \int_c^d \|\ddot\alpha(h(t))\| h'(t) dt & \text{if } h' \equiv 1 \\ \int_d^c \|\ddot\alpha(h(t))\| \cdot h'(t) dt & \text{if } h' \equiv -1 \end{cases}$
$= \int_a^b \ddot\alpha(u) du$, $u \triangleq h(t)$.
(b) By Ex 10.6. $l(N \circ \alpha) = \int_a^b \|\dot N \circ \alpha\| dt = \int_a^b \|-k\circ\alpha \cdot \dot\alpha\| dt = \int_a^b |k \circ \alpha| dt$.

11.11 (a) $d(f+g)(v) = \nabla(f+g) \cdot v = \nabla f \cdot v + \nabla g \cdot v = df(v) + dg(v)$ $\forall v \triangleq (p,v) \in R_p^{n+1}$, $p \in U$
(b) $d(fg)(v) = \nabla(fg) \cdot v = \nabla f(p) \cdot g(p) \cdot v + \nabla g(p) \cdot f(p) \cdot v$
So $dfg = g\,df + f\,dg$
(c) $d(h \circ f)(v) = \nabla(h \circ f) \cdot v = h'(f(p)) \nabla f(p) \cdot v$, So $d(h \circ f) = (h' \circ f) df$

11.12 (a) $\int_\alpha (x_2 dx_1 - x_1 dx_2) = \int_0^{2\pi} [2\sin t(-2\sin t) - 2\cos t(2\cos t)] dt = -8\pi$
(b) $\int_C (-x_2 dx_1 + x_1 dx_2) = \int_0^{2\pi} [(-b\sin t)(-a\sin t) + (a\cos t)(b\cos t)] dt = 2\pi ab$
(c) $\int_\alpha \sum_{i=1}^{n+1} x_i dx_i = f(\alpha(1)) - f(\alpha(0)) = \frac12(n+1)$, where $f(x) = \frac12 \sum_{i=1}^{n+1} x_i^2$: $df(v) = \nabla f(p) \cdot v = \sum_{i=1}^{n+1} p_i v_i$
$df(X_j) = p_j$. So $df = \sum_{i=1}^{n+1} x_i dx_i$

11.13 $w(\dot\alpha(t)) = \sum_{i=1}^{n+1} f_i(\alpha(t)) \dot\alpha_i(t) = \sum_{i=1}^{n+1} f_i(\alpha(t)) \cdot \frac{dx_i}{dt}$ So $\int_\alpha w = \int_a^b \sum_{i=1}^{n+1} (f_i \circ \alpha) \frac{dx_i}{dt} dt$

11.14 If C is connected, then there is a one-to-one, global parametrization $\alpha(t): [a,b] \to C$.
$\int_C w_X = \int_a^b X(\alpha(t))\cdot\dot\alpha(t)\,dt = \int_a^b \|\dot\alpha(t)\|dt = l(C)$ (as X is rotating $\nabla f/\|\nabla f\|$ by $-\pi/2$)
If C is not connected, then the above is true for each segment, so globally holds too

11.15 Treat $\dot\alpha$ as α, then $\dot\alpha(t) = (\cos\theta(t), \sin\theta(t))$ by proof in Thm 3 ($\theta(t) \triangleq \theta_0 + \int_{t_0}^t \eta(\beta(\tau))d\tau$)
As for uniqueness: If $\theta_1(t)$ and $\theta_2(t)$ satisfy: $\cos\theta_1(t) \equiv \cos\theta_2(t)$, $\sin\theta_1(t) \equiv \sin\theta_2(t)$
$\theta_1(t_0) = \theta_2(t_0) = \theta_0$, then $\theta_1(t) = \theta_2(t)$ for all $t \in I$. Proof: By first two equations
$\sin(\theta_1(t) - \theta_2(t)) = 0$ So $\cos(\theta_1(t) - \theta_2(t))\cdot(\dot\theta_1(t) - \dot\theta_2(t)) = 0$.
But $\sin(\theta_1 - \theta_2) = 0 \Rightarrow \cos(\theta_1 - \theta_2) \neq 0$, So $\dot\theta_1(t) - \dot\theta_2(t) = 0$ So $\theta_1(t) \equiv \theta_2(t)$ as it holds for $t=0$.

11.16 Let $\beta(t) = f(t)\cdot\alpha(t)$. Define $\varphi_1(t) = \varphi_1(a) + \int_{\alpha_t}\eta$, $\varphi_2(t) = \varphi_2(a) + \int_{\beta_t}\eta$
$\varphi_1(a)$ is chosen so that $\alpha(a)/\|\alpha(a)\| = (\cos\varphi_1(a), \sin\varphi_1(a))$ and $\varphi_1(a) \in [0, 2\pi)$
$\varphi_2(a)$ - - - - - - $\beta(a)/\|\beta(a)\| = (\cos\varphi_2(a), \sin\varphi_2(a))$ and $\varphi_2(a) \in [0, 2\pi)$
As $\beta(a)/\|\beta(a)\| = \alpha(a)/\|\alpha(a)\|$ (since $f>0$) and such choice of $\varphi_1(a), \varphi_2(a)$ is unique, we have
$\varphi_1(a) = \varphi_2(a)$. Furthermore, by proof in Thm 3,
$\alpha(t)/\|\alpha(t)\| = (\cos\varphi_1(t), \sin\varphi_1(t))$. $\beta(t)/\|\beta(t)\| = (\cos\varphi_2(t), \sin\varphi_2(t))$
As $\alpha(t)/\|\alpha(t)\| \equiv \beta(t)/\|\beta(t)\|$. $\cos\varphi_1(t) \equiv \cos\varphi_2(t)$, $\sin\varphi_1(t) \equiv \sin\varphi_2(t)$
and $\varphi_1(a) = \varphi_2(a)$. Same as the proof of uniqueness in Ex 11.15 we have
$\varphi_1(t) \equiv \varphi_2(t)$, $k(\alpha) = \frac{1}{2\pi}(\varphi_1(b) - \varphi_1(a)) = \frac{1}{2\pi}(\varphi_2(b) - \varphi_2(a)) = k(\beta)$. (now may need piece-wise, but still true)
Let $f = \|\alpha\|^{-1}$ (As $\|\alpha\| \neq 0$), then $k(\alpha) = k(\alpha/\|\alpha\|)$.
Actually no need of α being closed and $f(a) = f(b)$. $\int_\alpha \eta \equiv \int_\beta \eta$.

11.17 Since by Ex 11.16, α and $\alpha/\|\alpha\|$ have the same winding number, it is now equivalent to proving that with $\varphi(t,u)$ redefined as $\hat\varphi(t,u) = \varphi(t,u)/\|\varphi(t,u)\|$, the result holds. Now $\|\hat\varphi_u(t)\| = 1$ for all u, and t, and $\varphi(t,u)/\|\varphi(t,u)\|$ is continuous as $\|\varphi(t,u)\|$ is continuous and $\hat\varphi_u(t)$ is smooth on each $[t_i, t_{i+1}]$, $\hat\varphi_u(a) = \hat\varphi_u(b)$.
As $[a,b]\times[0,1]$ is compact, and $\hat\varphi$ is continuous, $\hat\varphi$ must be uniform continuous, i.e.
$\forall \varepsilon_1 > 0 \exists \delta_1 > 0 \forall \|(t_1,u_1) - (t_2,u_2)\| < \delta_1$ $\|\hat\varphi(t_1,u_1) - \hat\varphi(t_2,u_2)\| < \varepsilon_1$. Specifically, let $t_1 = t_2$
$\|\hat\varphi(t,u_1) - \hat\varphi(t,u_2)\| < \varepsilon_1$ i.e. $\hat\varphi(t,u_1)\cdot\hat\varphi(t,u_2) \geq 1 - \frac{\varepsilon_1^2}{2} = 1 - \varepsilon_2$ ($\varepsilon_2 \triangleq \frac{\varepsilon_1^2}{2}$) $\forall |u_1 - u_2| < \varepsilon_1$ (*)
Define $\theta_u(t) = \theta_u(a) + \int_{\hat\varphi_u}\eta$, $\theta_x(t) = \theta_x(a) + \int_{\hat\varphi_x}\eta$. $\forall u \in [0,1]$, $x \in (u-\varepsilon_1, u+\varepsilon_1)\cap[0,1]$
$\theta_u(a)$ is chosen so that $\hat\varphi_u(a) = (\cos\theta_u(a), \sin\theta_u(a))$. Likewise, $\hat\varphi_x(a) = (\cos\theta_x(a), \sin\theta_x(a))$
and $\theta_u(a), \theta_x(a) \in [0, 2\pi)$ unique! For $\forall t$, by proof in Thm 3,
$\hat\varphi_u(t) = (\cos\theta_u(t), \sin\theta_u(t))$, $\hat\varphi_x(t) = (\cos\theta_x(t), \sin\theta_x(t))$

So $\hat{\varphi}_u(t) \cdot \hat{\varphi}_x(t) = \cos(\theta_u(t) - \theta_x(t))$. By (*) $\cos(\theta_u(t) - \theta_x(t)) > 1 - \varepsilon_2$ (**)

Let $\theta_0 = \arccos(1-\varepsilon_2)$. ~~As $\theta_u(a), \theta_x(a) \in [0, 2\pi)$, $|\theta_u(a) - \theta_x(a)| < \theta_0$~~

So for $\forall t$, $\exists k_t \in \mathbb{Z}$, s.t. $|\theta_u(t) - \theta_x(t)| \in (2k_t\pi - \theta_0, 2k_t\pi + \theta_0)$

If $\exists t_1, t_2$, $t_1 < t_2$, $k_{t_1} \neq k_{t_2}$, $|\theta_u(t_1) - \theta_x(t_1)| \in (2k_{t_1}\pi - \theta_0, 2k_{t_1}\pi + \theta_0)$

$|\theta_u(t_2) - \theta_x(t_2)| \in (2k_{t_2}\pi - \theta_0, 2k_{t_2}\pi + \theta_0)$ Note $\theta_0 \in (0, \pi/2]$

As $\theta_u(t) - \theta_x(t)$ is continuous with respect to t, there must exist $t_3 \in (t_1, t_2)$

s.t. $\theta_u(t_3) - \theta_x(t_3) = 2\pi \cdot \min(k_{t_1}, k_{t_2}) + \pi$, ~~if ε_2 is small enough and thus θ_0 is small enough~~. But $\cos(\theta_u(t_3) - \theta_x(t_3)) = -1$ violating (**). Thus there is a $k \in \mathbb{Z}$,

s.t. $\forall t \in [a,b]$, $|\theta_u(t) - \theta_x(t)| \in (2k\pi - \theta_0, 2k\pi + \theta_0)$

But $|k(\varphi_u) - k(\varphi_x)| = \frac{1}{2\pi} |(\theta_u(b) - \theta_u(a)) - (\theta_x(b) - \theta_x(a))|$

~~$\leq \frac{1}{2\pi}(|\theta_u(b) - \theta_x(b)| + |\theta_u(a) - \theta_x(a)|)$~~

$= \frac{1}{2\pi} |(\theta_u(b) - \theta_x(b)) - (\theta_u(a) - \theta_x(a))|$

$< \frac{1}{2\pi} 2\theta_0 = \frac{\theta_0}{\pi}$ ~~$\pi/3$ is to make θ_0 sufficiently small to ensure $\exists t_3$~~

So $\forall u \in [a,b]$ $\forall \varepsilon \in (0, 1/2)$, $\exists \varepsilon_2 = 1 - \cos(\varepsilon\pi)$, $\varepsilon_1 = \sqrt{2\varepsilon_2}$, s.t. $\forall x \in (u-\varepsilon_1, u+\varepsilon_1) \cap [0,1]$,

$|k(\varphi_x) - k(\varphi_u)| < \varepsilon$. So $k(\varphi_u)$ is a continuous function of u

Finally, as $k(\varphi_u)$ can only assume integer value, $k(\varphi_0) = k(\varphi_1)$

Note: φ can be on any $[a,b] \times [c,d]$ ($a, b, c, d \neq \infty$) and $k(\varphi_c) = k(\varphi_d)$

11.18 (a) $\forall n$. define $\alpha(t) = (\cos nt, \sin nt)$, i.e. $\alpha'(t) = (\frac{1}{n}\sin nt, \frac{-1}{n}\cos nt)$

Then following Example 2 on Pg 75, $\int_{\dot\alpha} \eta = n \int_0^{2\pi} \frac{1 \cdot 1}{\cos^2 t + \sin^2 t} dt = 2n\pi$

i.e. the rotation index of α is n.

(b) We follow the definitions of φ, ψ, ϕ as in the hint, but define to more formally

Let $u \in \mathbb{R}^2$, $u \neq 0$ define $h(t) = \alpha(t) \cdot u$. Since α is compact, h must attain its minimum, say, at t_0. ~~By Lagrange multiplier Thus, ... $u = \lambda$~~

So $h'(t_0) = \dot\alpha(t_0) \cdot u = 0$, i.e. $\dot\alpha(t_0) \perp u$. By definition, ϕ is continuous.

$k(\varphi_0)$ is the rotation index of α, because $\varphi_0(t) = \psi(t,t) = \dot\alpha(t)/\|\dot\alpha(t)\|$.

As for $k(\varphi_1)$, when $t \in (t_0, t_0 + \tau/2]$ $\varphi_1(t) = \psi(t_0, t_0 + 2t) = (\alpha(t_0 + 2t) - \alpha(t_0))/\|\alpha(t_0+2t) - \alpha(t_0)\|$ $t_0 \to t_0 + \tau$

Now the η is exact because $\varphi_1(t) \cdot u \geq 0$ and We can set $v = -u$, and have ~~$\theta_v = \theta_u$~~

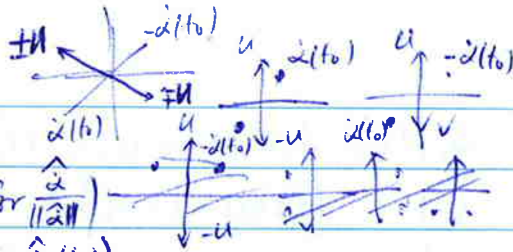
$V = \mathbb{R}^2 - \{rv \mid r > 0\}$, η is exact on V. Here $\varphi_1(t) \cdot u \geq 0$, because $\forall t$

$h(t) \geq h(t_0) \Rightarrow (\alpha(t) - \alpha(t_0)) \cdot u \geq 0 \Rightarrow \varphi_1(t) \cdot u \geq 0$ $(t \in (t_0, t_0 + \tau/2))$

So $\int_{\varphi_1|(t_0, t_0+\tau/2]} \eta = \theta_v(\varphi_1(t_0 + \frac{\tau}{2})) - \theta_v(\varphi_1(t_0)) = \theta_v(-\dot\alpha(t_0)/\|\dot\alpha(t_0)\|) - \theta_v(\dot\alpha(t_0)/\|\dot\alpha(t_0)\|)$

$= \pm\pi$

For $t \in (t_0 + \frac{\tau}{2}, t_0 + \tau]$, $\varphi_1(t) = \psi(2t - t_0 - \tau, t_0 + \tau) = (\alpha(t_0) - \alpha(2t - t_0 - \tau))/\|\cdot\|$ $t_0 \to t_0 + \tau$



Now we set $V' = u$, $V' = \mathbb{R}^2 - \{rv \mid r > 0\}$. η is exact on V'

$\int_{\varphi_1|(t_0+T/2,\, t_0+T]} \eta = \theta_{V'}(\hat{\dot\alpha}(t_0)) - \theta_{V'}(-\hat{\dot\alpha}(t_0))$ ($\hat{\dot\alpha}$ stands for $\frac{\dot\alpha}{\|\dot\alpha\|}$)

But for $\forall u$, $\theta_u(-\hat{\dot\alpha}(t_0)) - \theta_u(\hat{\dot\alpha}(t_0)) + \theta_{-u}(\hat{\dot\alpha}(t_0)) - \theta_{-u}(-\hat{\dot\alpha}(t_0))$

$= \pm 2\pi$ (if $-u \searrow +u$, then 2π, if $-u \nwarrow +u$, then -2π)

Thus, $k(\varphi_0) = k(\varphi_1) = \pm 1$ i.e. rotation index is ± 1.

11.19 (a) $h'(t_0) = 0 \iff \dot\alpha(t_0)\cdot u = 0 \iff \dot\alpha(t_0) \perp u$ } $\iff N(\alpha(t_0)) = \pm u$. Let $N(\alpha(t_0)) = \delta u$
Since $\dot\alpha(t_0) \perp N_{(\alpha(t_0))}$ $\quad \delta = \pm 1$.

$h''(t_0) = \ddot\alpha(t_0)\cdot u = \ddot\alpha \cdot \delta N = k\cdot\delta = k(\alpha(t_0)) \cdot N(\alpha(t_0))\cdot u$.

(b) construct $\theta(t)$ as in Ex 11.15. By Ex 10.10 $\frac{d\theta}{dt} = k\circ\alpha$. Then rotation index is
$\frac{1}{2\pi}\int_{\dot\alpha}\eta = \frac{1}{2\pi}(\theta(t_0+\tau) - \theta(t_0)) = \frac{1}{2\pi}\int_{t_0}^{t_0+\tau}\frac{d\theta}{dt}dt = \int_{t_0}^{t_0+\tau}(k\circ\alpha)(t)dt$

Gauss map N of C is onto because: $\forall u \in S^1$, $h(t_1) = h(t_1+\tau)$ $\forall t_1, t_1+\tau \in \mathbb{R}$, τ is period
so there must be $t_0 \in (t_1, t_1+\tau)$ s.t. $h'(t_0) = 0$ So $N(\alpha(t_0)) = \pm u$
Since $\alpha(t)$ is periodic $h(t)$ must have both ~~maximum~~ minimum and ~~minimum~~ maximum say, t_0, t_0' resp.
$h'(t_0) = h'(t_0') = 0$, $h''(t_0) \geq 0$ $h''(t_0') \leq 0$. But since $N = \pm u$, $N\cdot u \neq 0$ So $h''(t_0) \neq 0$
So $h''(t_0) > 0$. Likewise, $h''(t_0') < 0$. $h''(t_0) > 0 \Rightarrow u\cdot N(\alpha(t_0)) > 0 \Rightarrow N(\alpha(t_0)) = u$
$h''(t_0') < 0 \Rightarrow u\cdot N(\alpha(t_0')) < 0 \Rightarrow N(\alpha(t_0')) = -u$. So N is onto

(c) As $k > 0$ $\int_{t_0}^{t}(k\circ\alpha)(t)dt$ monotonically increasing wrt t. Set $\theta(t) = \theta_0 + \int_{t_0}^{t}\eta(\dot\alpha(\tau))d\tau$
then $\dot\alpha = (\alpha\cdot \cos\theta(t), \sin\theta(t))$ As $N(c) = N(t_0)$, so $(\cos\theta(c), \sin\theta(c)) = (\cos\theta_0, \sin\theta_0)$
So $\theta(c) = 2n\pi + \theta_0$. But by Ex 10.10, $\int_{t_0}^{c}(k\circ\alpha)(t)dt = \int_{t_0}^{c}\frac{d\theta}{dt}dt = \theta(c) - \theta_0 = 2n\pi$
As $k > 0$, $c > t_0$ so $n > 0$. If $n = 2$ then there is a $t_1 \in (t_0, c)$: $\theta(t_1) = \theta_0 + 2\pi$
because $\theta(t)$ is continuous. But $(\cos\theta(t_1), \sin\theta(t_1)) = (\cos\theta_0, \sin\theta_0)$, So $N(\alpha(t_1)) = N(\alpha(t_0))$
But ~~by~~ that contradicts with the assumption that $N(t) \neq N(t_0)$ $\forall t \in (t_0, c)$
So $n = 1$, i.e. $\int_{t_0}^{c}(k\circ\alpha)(t)dt = 2\pi$.
By ~~definition~~ (b) $\int_{t_0}^{t_0+\tau}(k\circ\alpha)(t)dt$ = rotation index of $\alpha * 2\pi = \pm 2\pi$.
As $k > 0$. It equals 2π, which in turn equals $\int_{t_0}^{c}(k\circ\alpha)(t)dt$.
As $k > 0$ $c = t_0 + \tau$. (a) has shown the Gaussian map is onto,
Now we've proven that $N(t) = N(t_0)$ iff $t = t_0 + \tau\cdot n$. But τ is period of α.
So Gaussian map is injection, In sum, it is one-to-one

11.20 (a) α_f is just one point a_0, so obviously $k(f) = 0$ (construct $v = -a_0$, θ_v)
(b) $\alpha_f(t) = (a_n \cos nt, a_n \sin nt)$ Similar to Example 2 on Pg. 75, $k(f) = n$.
(c) Construct $\varphi: [0, 2\pi] \times [0,1] \to \mathbb{R}^2 - \{0\}$. $\varphi(t,u) = f(u(\cos t + i\sin t))$
↑ by $f(z) \neq 0$ $\forall z: |z| \leq 1$ $\quad$ obviously continuous

$\varphi_0(t) = f(0) \neq 0$, So $k(\varphi_0(t)) = 0$.

$\varphi_1(t) = f(\cos t + i\sin t)$ $\quad k(f) = k(\varphi_1(t)) \overset{*}{=} k(\varphi_0(t)) = 0$ (*by Ex 11.17)

(d) Construct $\varphi(t,u) = \begin{cases} u^n f\left(\frac{1}{u}(\cos t + i\sin t)\right) & \text{if } u \neq 0 \\ a_n(\cos t + i\sin t)^n & \text{if } u = 0. \end{cases}$ on $[0,2\pi]\times[0,1]$

By def of φ and $f(z) \neq 0$ $\forall z \in \mathbb{C}, |z| \geq 1$, we have $\varphi(t,u) \not\equiv 0$.

As $\lim_{u\to 0} u^n f\left(\frac{1}{u}(\cos t + i\sin t)\right) = \lim_{u\to 0} u^n \sum_{k=0}^{n} a_n(\cos kt + i\sin kt)\frac{1}{u^k} = a_n(\cos nt + i\sin nt)$

$= \varphi(t,0)$ So φ is continuous

$\varphi(t,0) = a_n(\cos nt + i\sin nt)$. By Example 2 on Pg 75. $k(\varphi(t,0)) = n$

By Ex 11.17. $k(f) = k(\varphi(t,1)) = k(\varphi(t,0)) = n$.

(e). Combining (c),(d), (c) says $k(f) = 0$, (d) says $k(f) = n$. So ~~either~~ $n = 0$

11.21 If no point of $\alpha(t)$ lies on positive x_1-axis, then choose $v = (1,0)$, by θ_v, we have $k(\alpha) = 0$, correct.
Else Let $a < t_1 < t_2 < \cdots < t_m < b$ be the set of all $t \in (a,b)$ such that $\alpha(t)$ lies on the positive x_1-axis. (we will discuss $t = a, b$ specially) ~~Note as $\alpha(a) = \alpha(b)$, even if $\alpha(a)$ is not on positive x_1-axis, we can still reparametrize $\alpha(t)$ into $\beta(t) = \alpha(t_1 + t - a)$, then $\beta(a) = \alpha(t_1)$ which is on x_1-axis.~~ Denote $t_0 = a$, $t_{m+1} = b$. For all $i = 1, 2, \cdots, m$, if $\alpha(t_i)$ crosses positive x_1-axis upward, define $\delta_i = 1$. If crosses downward, define $\delta_i = -1$. If $\alpha(a) = \alpha(b)$ is on positive x_1-axis, define $\delta_0 = \delta_{m+1}$ likewise in $\{\pm 1\}$. ~~If $\alpha(a)$ is not on positive x_1-axis, define $\delta_0 = \delta_{m+1} = 0$~~.

$k(\alpha) = \frac{1}{2\pi}\sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \eta(\dot\alpha(t))\,dt = \frac{1}{2\pi}\sum_{i=0}^{m} \lim_{\varepsilon\to 0^+} \int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon} \eta(\dot\alpha(t))\,dt = \frac{1}{2\pi}\sum_{i=0}^{m} \lim_{\varepsilon\to 0^+} \int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon} d\theta_v(\alpha(t))\,dt$

where $v = (1,0)$ and θ_v is defined as in proof of Thm 3. We check two consecutive crossings of positive x_1-axis: $(i = 1, \cdots, m-1)$:

angle means $\lim_{\varepsilon\to 0^+}[\theta_v(\alpha(t_{i+1}-\varepsilon)) - \theta_v(\alpha(t_i+\varepsilon))]$.

i	$i+1$	δ_i	δ_{i+1}	angle	formula
↗	↗	1	+1	2π	
↗	↘	1	-1	0	$(\delta_i + \delta_{i+1})\cdot\frac{2\pi}{2}$
↘	↘	-1	-1	-2π	
↘	↗	-1	1	0	

So $k(\alpha) = \frac{1}{2\pi}\sum_{i=1}^{m-1}(\delta_i + \delta_{i+1})$

if $\alpha(a)$ is not on pos x_1-axis:

$+ \frac{1}{2\pi}\left(\theta_v(\alpha(b)) - 2\pi\cdot\frac{1-\delta_m}{2}\right)$

$+ \frac{1}{2\pi}\left(2\pi\frac{1+\delta_1}{2} - \theta_v(\alpha(b))\right)$

$= \sum_{i=1}^{m} \delta_i$

If $\alpha(a)$ is on pos x_1-axis, then $k(\alpha) = \frac{1}{2}\sum_{i=0}^{m}(\delta_i + \delta_{i+1}) = \sum_{i=0}^{m}\delta_i$ (as $\delta_{m+1} = \delta_0$)

So the conclusion is correct in both cases.

11.22 (a) let $\beta(t) = \alpha(t) - p$

$\eta(\dot\beta) = -\frac{\beta_2}{\beta_1^2+\beta_2^2}dx_1 + \frac{\beta_1}{\beta_1^2+\beta_2^2}dx_2 = -\frac{(\alpha_2(t)-b)\,dx_1}{(\alpha_1(t)-a)^2+(\alpha_2(t)-b)^2} + \frac{(\alpha_1(t)-a)\,dx_2}{(\alpha_1(t)-a)^2+(\alpha_2(t)-b)^2}$

So $k(\beta) = \int_\beta \eta(\beta) = \int_\alpha k_p(\alpha)$. We know $k(\beta) = 2k\pi$ $k \in \mathbb{Z}$, so $\frac{1}{2\pi}\int_\alpha k_p(\alpha)$ is integer

(b) Suppose p and q are joined by continuous $\beta : [c,d] \to \mathbb{R}^2 - \text{Image }\alpha$ s.t. $\beta(c) = p$. $\beta(d) = q$

Define $\varphi(t,u) = \alpha(t) - \beta(u)$ on $[a,b]\times[c,d] \to \mathbb{R}^2 - \{0\}$ (Since $\beta \to \mathbb{R}^2 - \text{Image }\alpha$ so $\varphi \neq 0$)

Obviously, φ is continuous. $\varphi(t,c) = \alpha(t) - p$. $\varphi(t,d) = \alpha(t) - q$

$k(\varphi(t,c)) = k(\varphi(t,d)) \Rightarrow k_p(\alpha) = k_q(\alpha)$ (as $k_p(\alpha) = k(\alpha(t) - p)$)

12.1 The matrix corresponding to L_p is $A = \|g\|^{-3}(g \cdot g^T - \|g\|^2 I)H$ ($g = \nabla f$, H is the Hessian)

$L_p(v) = -\|g\|^{-1} H \cdot v$, $k(v) = -\|g\|^{-1} v^T H v = \varphi_p(v)$ for $v \in S_p$.

12.2 $\nabla f = (1,1,\dots,1)$, $v_i = \frac{1}{\sqrt{2}}(1,0\dots0,-1,0\dots0)$ where -1 is the i^{th} spot after 1, $i = 1\dots n$.

$\nabla f = \sqrt{n+1}$, $\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$. So $k(v) = \varphi_p(v) = 0$ $\forall v$ by Ex 12.1 So any $v \in S_p$ $\|v\| = 1$ is a principle curvature direction, with principle curvature 0.

$K(p) = 0$ $H(p) = 0$.

12.3 $\nabla f = (2x_1, \dots, 2x_{n+1})$ $\|\nabla f(p)\| = 2r$ $H = 2 \cdot I$. $k(v) = \frac{-1}{r} v \cdot v$.

Any $v \in S_p$ $\|v\| = 1$ is a principle curvature direction, with principle curvature $\frac{-1}{r}$.

$K(p) = (-r)^{-n}$, $H(p) = \frac{-1}{r}$

12.4 $\nabla f = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}, \frac{2x_3}{c^2}\right)$ $H = \begin{pmatrix} 2a^{-2} & 0 & 0 \\ 0 & 2b^{-2} & 0 \\ 0 & 0 & 2c^{-2} \end{pmatrix}$ $\|\nabla f(p)\| = \frac{2}{a}$ $\nabla f(p) = (\frac{2}{a}, 0, 0)$

$k(v) = -\frac{a}{2}\left(\frac{2}{a^2}v_1^2 + \frac{2}{b^2}v_2^2 + \frac{2}{c^2}v_3^2\right) = -a\left(\frac{v_1^2}{a^2} + \frac{v_2^2}{b^2} + \frac{v_3^2}{c^2}\right)$ $\forall v \in S_p$: $v = (0, v_2, v_3)$

$v_2^2 + v_3^2 = 1$, So $k(v)$ attains its extremum at $(0, \pm1, 0)$ and $(0,0,\pm1)$ i.e. principle curvature directions, corresponding to principal curvature $\frac{-a}{b^2}$ and $\frac{-a}{c^2}$ respectively.

$K(p) = a^2/b^2c^2$, $H(p) = \frac{-a}{2}\left(\frac{1}{b^2} + \frac{1}{c^2}\right)$

12.5 $\nabla f = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}, -\frac{2x_3}{c^2}\right)$ $H = 2\begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & -c^{-2} \end{pmatrix}$ $\|\nabla f(p)\| = \frac{2}{a}$ $\nabla f(p) = (\frac{2}{a}, 0, 0)$

$k(v) = -\frac{a}{2}\left(\frac{2}{a^2}v_1^2 + \frac{2}{b^2}v_2^2 - \frac{2}{c^2}v_3^2\right)$ $\forall v \in S_p$. $v = (0, v_2, v_3)$ $v_2^2 + v_3^2 = 1$ so

$k(v) = \frac{-a}{2}\left(\frac{2}{b^2}v_2^2 - \frac{2}{c^2}v_3^2\right) = -a\left(\frac{1}{b^2}(1 - v_3^2) - \frac{1}{c^2}v_3^2\right) = a\left[\left(\frac{1}{b^2} + \frac{1}{c^2}\right)v_3^2 - \frac{1}{b^2}\right]$ $v_3^2 \in [0,1]$

$k(v)$ attains max when $v_3^2 = 1$, max $= \frac{a}{c^2}$, attains min when $v_3^2 = 0$ min $= \frac{-a}{b^2}$

So principal curvature and principle curvature directions are: $((0,0,\pm1), \frac{a}{c^2})$, $((0,\pm1,0), \frac{-a}{b^2})$

$K(p) = -\frac{a^2}{b^2c^2}$, $H(p) = \frac{a}{2}\left(\frac{1}{c^2} - \frac{1}{b^2}\right)$

12.6 $\nabla f = \left(2x_1, 2x_2(1 - 2(x_2^2 + x_3^2)^{-\frac{1}{2}}), 2x_3(1 - 2(x_2^2 + x_3^2)^{-\frac{1}{2}})\right)$

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 - 4x_3^2(x_2^2 + x_3^2)^{-3/2} & 4x_2x_3(x_2^2 + x_3^2)^{-3/2} \\ 0 & 4x_2x_3(x_2^2 + x_3^2)^{-3/2} & 2 - 4x_2^2(x_2^2 + x_3^2)^{-3/2} \end{pmatrix}$$

For (a) $\nabla f(p) = (0,2,0)$ $\|\nabla f\| = 2$, ($p = (0,3,0)$)

$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}$

(b) $p = (0,1,0)$, $\nabla f(p) = (0,-2,0)$ $\|\nabla f\| = 2$

$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

(a) $k(v) = -V_1^2 - V_2^2 - \frac{1}{3}V_3^2$, $V = (V_1, 0, V_3)$ $V_1^2 + V_3^2 = 1$, $k(v) = -1 + \frac{2}{3}V_3^2$

min $= -1$ when $V_3 = 0$. max $= -\frac{1}{3}$ when $V_3 = \pm 1$. So $((\pm 1, 0, 0), -1)$ and $((0, 0, \pm 1), -\frac{1}{3})$

$H(p) = -\frac{2}{3}$, $K(p) = \frac{1}{3}$

(b) $k(V) = -V_1^2 - V_2^2 + V_3^2$ $V = (V_1, 0, V_3)$ $V_1^2 + V_3^2 = 1$ $k(V) = -1 + 2V_3^2$

min $= -1$ when $V_3 = 0$., max $= 1$ when $V_3 = \pm 1$

So $((\pm 1, 0, 0), -1)$, $((0, 0, \pm 1), 1)$ $H(p) = 0$. $K(p) = -1$

12.7 If (λ_i, V_i) are eigenvalue vector of L_p for S. For $\forall V \in S_p$, $L_p(V) = -\nabla_V N = -(-\nabla_V(-N)) = -\tilde{L}_p(V)$ where $\tilde{L}_p$ stands for the Waingarten map for orientation $-N$. Thus Specifically $L_p(V_i) = \lambda_i V_i \Leftrightarrow \tilde{L}_p(V_i) = -\lambda_i V_i$. So L_p's eigenvalue λ_i corresponds to $\tilde{L}_p$'s eigenvalue $-\lambda_i$. So $\tilde{K} = \prod_{i=1}^{n}(-\lambda_i) = (-1)^n \prod_{i=1}^{n} \lambda_i = (-1)^n K$

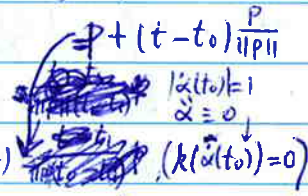
12.8 As $n=2$, the Gaussian curvature is independent of orientation

Apply Thm 5. $Z = \frac{1}{2}\nabla f(p) = (p, x_1, x_2, -x_3)$ take $V_1 = (p, x_3, 0, x_1)$, $V_2 = (0, x_3, x_2)$

So $V_1 \perp Z$, $V_2 \perp Z$ $\det\begin{pmatrix}\nabla_{V_1}Z \\ \nabla_{V_2}Z \\ Z(p)\end{pmatrix} = \begin{vmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \\ x_1 & x_2 & -x_3 \end{vmatrix} = (x_1^2 + x_2^2 - x_3^2)x_3$

$\det\begin{pmatrix}V_1 \\ V_2 \\ Z(p)\end{pmatrix} = \begin{vmatrix} x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \\ x_1 & x_2 & -x_3 \end{vmatrix} = -x_3(x_1^2 + x_2^2 + x_3^2)$, $\|Z(p)\| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

So $K(p) = x_3(x_1^2 + x_2^2 - x_3^2) / [(x_1^2 + x_2^2 + x_3^2) \cdot (-x_3)(x_1^2 + x_2^2 + x_3^2)] = 0$



This is because through each point p, there's a $\alpha(t)$ ($k(\dot{\alpha}(t_0)) = 0$) which lie completely in S, so S doesn't force any acceleration. Besides, if S is oriented outward, then S always bends away from N, so $k(V) \le 0$. If oriented inward, then $k(V) \ge 0$. In whatever case, 0 is an extreme point of $k(V)$. So 0 is an eigenvalue of L_p. So $K(p) = 0$.

12.9 $Z = \frac{1}{2}\nabla f(p) = (p, x_1/a^2, x_2/b^2, -x_3/c^2)$ For $x_3 \ne 0$ we may take

$V_1 = (p, x_3/c^2, 0, x_1/a^2)$, $V_2 = (p, 0, x_3/c^2, x_2/b^2)$ $V_1, V_2 \perp Z$

$\det\begin{pmatrix}\nabla_{V_1}Z \\ \nabla_{V_2}Z \\ Z(p)\end{pmatrix} = \begin{vmatrix} x_3/a^2c^2 & 0 & -x_1/a^2c^2 \\ 0 & x_3/b^2c^2 & -x_2/b^2c^2 \\ x_1/a^2 & x_2/b^2 & -x_3/c^2 \end{vmatrix} = \frac{x_3}{a^2b^2c^4}(x_1^2/a^2 + x_2^2/b^2 - x_3^2/c^2) = \frac{x_3}{a^2b^2c^4}$

$\det\begin{pmatrix}V_1 \\ V_2 \\ Z(p)\end{pmatrix} = \begin{vmatrix} x_3/c^2 & 0 & x_1/a^2 \\ 0 & x_3/c^2 & x_2/b^2 \\ x_1/a^2 & x_2/b^2 & -x_3/c^2 \end{vmatrix} = \frac{-x_3}{c^2}\left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)$

$\|Z(p)\| = \left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^{1/2}$

$K(p) = -\left[a^2b^2c^2\left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^2\right]^{-1}$. negtitive. At each point p, there are some directions bends towards N, some directions bending away from N. So the max of $k(v) > 0$, min $k(v) < 0$

As $K(p)$ = product of two extreme values, $K(p) < 0$

12.10 $Z = \frac{1}{2}\nabla f(p) = (p, \frac{2}{a^2}x_1, \frac{2}{b^2}x_2, -1)$, $V_1 = (p, 1, 0, \frac{2}{a^2}x_1)$ $V_2 = (p, 0, 1, \frac{2}{b^2}x_2)$

$\det\begin{pmatrix}\nabla_{v_1}Z\\ \nabla_{v_2}Z\\ Z(p)\end{pmatrix} = \begin{vmatrix}2/a^2 & 0 & 0\\ 0 & 2/b^2 & 0\\ 2x_1/a^2 & 2x_2/b^2 & -1\end{vmatrix} = -4/a^2b^2$ $\det\begin{pmatrix}V_1\\ V_2\\ Z(p)\end{pmatrix} = \begin{vmatrix}1 & 0 & 2x_1/a^2\\ 0 & 1 & 2x_2/b^2\\ 2x_1/a^2 & 2x_2/b^2 & -1\end{vmatrix} = -1-4x_1^2/a^4-4x_2^2/b^4$ $\|Z(p)\|^2 = 1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4}$

$k(p) = 4/a^2b^2(1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4})^2$. $k(p)>0$ As can be seen from the fact that S bends towards N at all points in all directions in S_p if S is inward oriented. If outward, then always bend away from N in all directions. So the product $k(p)>0$.

12.11 $Z=(p, \frac{2x_1}{a^2}, \frac{-2x_2}{b^2}, -1)$, $V_1=(p,1,0,\frac{2x_1}{a^2})$, $V_2=(p,0,1,\frac{-2x_2}{b^2})$

$\det\begin{pmatrix}\nabla_{v_1}Z\\ \nabla_{v_2}Z\\ Z(p)\end{pmatrix} = \begin{vmatrix}2/a^2 & 0 & 0\\ 0 & -2/b^2 & 0\\ 2x_1/a^2 & -2x_2/b^2 & -1\end{vmatrix} = 4/a^2b^2$, $\det\begin{pmatrix}V_1\\ V_2\\ Z(p)\end{pmatrix} = \begin{vmatrix}1 & 0 & 2x_1/a^2\\ 0 & 1 & -2x_2/b^2\\ 2x_1/a^2 & -2x_2/b^2 & -1\end{vmatrix} = -1-4x_1^2/a^4-4x_2^2/b^4$ $\|Z(p)\|^2 = 1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4}$

$k(p) = -4/a^2b^2(1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4})^2 < 0$ hard to plot and analyze its shape but look at the graph at ~~http://here~~ http://users.rsise.anu.edu.au/~xzhang/reading/ex1211.jpg

12.12 (a) Cylinder C: $g(x_1,x_2,x_3) = f(x_1,x_2)$ $= f(x,y)$, $Z = \nabla g(p) = (f'_x, f'_y, 0)$,
$V_1 = (0,0,1)$. $V_2 = (f'_y, f'_x, 0)$ $\nabla_{v_1}Z = (0,0,0)$, So $k(p)=0$ by Thm 5.
(b) $g(x_1\cdots x_{n+1}) = f(x_1\cdots x_n)$ $Z = \nabla g(p) = (f'_{x_1},\cdots,f'_{x_n},0)$
$V_1 = (0,\cdots,0,1)$ and then decide $V_2\cdots V_n$. $\nabla_{v_1}Z = (0,\cdots,0)$ So $k(p)=0$.

12.13 $f = x_{n+1} - g(x_1\cdots x_n)$ $Z = \nabla f(p) = (-g'_1,\cdots,-g'_n,1)$, (So $Z\cdot(0,0\cdots0,1)>0$).
$V_1 = (1,0,\cdots 0, g'_1),\cdots, V_n = (0\cdots 0,1,g'_n)$, $\nabla_{v_1}Z = (-g''_{11},\cdots,-g''_{n1},0)\cdots,\nabla_{v_n}Z = (-g''_{n1},\cdots,-g''_{nn},0)$
$\det\begin{pmatrix}\nabla_{v_1}Z\\ \nabla_{v_n}Z\\ Z(p)\end{pmatrix} = \begin{pmatrix}-g''_{11} & \cdots & -g''_{nn} & 0\\ \cdots & & & \\ -g''_{n1} & \cdots & -g_{nn} & 0\\ -g'_1 & \cdots & -g'_n & 1\end{pmatrix} = (-1)^n\det\left(\frac{\partial^2 g}{\partial x_i\partial x_j}\right)$, $\det\begin{pmatrix}V_1\\ V_n\\ Z(p)\end{pmatrix} = \begin{vmatrix}1 & 0 & \cdots & 0 & g'_1\\ & & 0 & & \vdots\\ 0 & & \cdots & 1 & g'_n\\ -g'_1 & \cdots & & -g'_n & 1\end{vmatrix} = 1+\sum_{i=1}^n\left(\frac{\partial g}{\partial x_i}\right)^2 = \|Z_p\|^2$ ← easy proof by induction

$k(p) = (-1)^n\cdot(-1)^n\det\left(\frac{\partial^2 g}{\partial x_i\partial x_j}\right)/\left(1+\sum_{i=1}^n\left(\frac{\partial g}{\partial x_i}\right)^2\right)^{\frac{n}{2}+1} = \det\left(\frac{\partial^2 g}{\partial x_i\partial x_j}\right)/\left(1+\sum_{i=1}^n\left(\frac{\partial g}{\partial x_i}\right)^2\right)^{1+n/2}$

12.14. ~~If $v\times w=0$ then $\exists\lambda\in R$ $v=\lambda w$, $L_p(v)\times L_p(w) = \lambda L_p(w)\times L_p(w) = 0 = k(p)\,v\times w$~~
Both $L_p(v)\times L_p(w)$ and ~~Otherwise~~ $v\times w\in S_p^\perp$ (even if $v\times w=0$ i.e. $v//w$). So to prove the result, one only needs to prove that $N(p)\cdot L_p(v)\times L_p(w) = N(p)\cdot v\times w$, where $N(p)$ is Gauss map. $\|N(p)\|=1$
By Thm 5, $k(p) = \begin{vmatrix}-L_p(v)\\ -L_p(w)\\ N(p)\end{vmatrix} / \|N(p)\|^2 \begin{vmatrix}v\\ w\\ N(p)\end{vmatrix}$ So
$N(p)\cdot L_p(v)\times L_p(w) = \begin{vmatrix}L_p(v)\\ L_p(w)\\ N(p)\end{vmatrix} = k(p)\cdot\begin{vmatrix}v\\ w\\ N(p)\end{vmatrix} = k(p)\,N(p)\cdot v\times w$.

12.15. By Thm 5. $k(p) = \begin{vmatrix}\nabla_v Z\\ \nabla_w Z\\ Z(p)\end{vmatrix} / \|Z(p)\|^2\begin{vmatrix}v\\ w\\ Z(p)\end{vmatrix} = Z(p)\cdot\nabla_v Z\times\nabla_w Z/\|Z(p)\|^4$
as $\begin{vmatrix}v\\ w\\ Z(p)\end{vmatrix} = Z(p)\cdot v\times w = Z(p)\cdot Z(p) = \|Z(p)\|^2$
$\begin{vmatrix}\nabla_v Z\\ \nabla_w Z\\ Z(p)\end{vmatrix} = Z(p)\cdot v\times w$

12.16 By Thm 2, the eigenvectors of L comprises an orthonormal basis for S_p. Let them be $(\alpha_1, \cdots \alpha_n)$, (Let $V=(V_1, \cdots V_n) = (\alpha_1 \cdots \alpha_n)T$ [A] with corresponding eigenvalues $k_1(p) \cdots k_n(p)$). By Thm 3, ~~$k(V_i) = \sum_{j=1}^n k_j(p) t_i$~~ As $V_i = \sum_{j=1}^n \alpha_j t_{ji}$, so $k(V_i) = \sum_{j=1}^n k_j(p) t_{ji}^2$

So $\sum_{i=1}^n k(V_i) = \sum_{i,j=1}^n k_i(p) t_{ij}^2 = \sum_{i=1}^n k_i(p) \cdot \sum_{j=1}^n t_{ij}^2$, As Both V and A are orthonormal, $I = V^T V = T^T A^T A T = T^T T$, So T is also orthonormal. So $TT^T = I$ (I is identity)

So $\sum_{j=1}^n t_{ij}^2 = 1$ for all $i = 1 \cdots n$. So $\sum_{i=1}^n k(V_i) = \sum_{i=1}^n k_i(p)$, thus $H(p) = \frac{1}{n}\sum_{i=1}^n k_i(p) = \frac{1}{n}\sum_{i=1}^n k(V_i)$

12.17 (a) Obvious by Thm 3. Anyway $L(v(\theta)) = (\cos\theta) L(v_1) + (\sin\theta) L(v_2)$

$k(v(\theta)) = L(v(\theta)) \cdot v(\theta) = \cos^2\theta \cdot L(v_1) \cdot v_1 + \sin^2\theta \cdot L(v_2) v_2$

$+ \sin\theta\cos\theta \left(L(v_1)\cdot v_2 + L(v_2)\cdot v_1 \right)$

$L(v_1)\cdot v_2 = k_1 \cdot v_1 \cdot v_2 = 0$. $L(v_2)\cdot v_1 = 0$.

So $k(v(\theta)) = \cos^2\theta \cdot k_1 + \sin^2\theta \cdot V_2$

(b) $H_p = \frac{1}{2}(k_1+k_2)$, $\frac{1}{2\pi}\int_0^{2\pi} k(v(\theta))\,d\theta = \frac{1}{2\pi}\left[k_1 \int_0^{2\pi}\cos^2\theta\,d\theta + k_2\int_0^{2\pi}\sin^2\theta\,d\theta\right] = \frac{1}{2}(k_1+k_2) = H_p$.

12.18 $\operatorname{div} N = \operatorname{tr}(v \mapsto \nabla_v N) = \operatorname{tr}(-L_p)$ ~~$= -\operatorname{tr}(L_p)$~~

If $v_1 \cdots v_n$ [$\lambda_1 \cdots \lambda_n$] are eigenvectors [values] of L_p, then $v_1, \cdots v_n$ [$-\lambda_1 \cdots -\lambda_n$] are eigenvectors [values] of $-L_p$ because

$L_p(v_i) = \lambda_i v_i \iff -L_p(v_i) = -\lambda_i v_i$ So $\operatorname{tr}(-L_p) = \sum_{i=1}^n (-\lambda_i) = -\operatorname{tr}(L_p)$

So $H_p = \frac{1}{n}\operatorname{tr}(L_p) = \frac{-1}{n}\operatorname{div} N$

12.19 (a) $\tilde{S}$ is $g^{-1}(c)$. $\nabla g(p) = 0 \iff \frac{1}{a}\nabla f(p/a) = 0$

But S is n-surface, so $\nabla f(p/a) \neq 0$ for all p and thus $\nabla g(p) \neq 0 \ \forall p$, so $\tilde{S}$ is n-surface

$p \in S \iff f(p) = c \iff g(ap) = f(p) = c \iff ap \in \tilde{S}$

(b) $\forall N$ in the Gauss image of S, $\exists p$, s.t. $\nabla f/\|\nabla f\|\big|_p = N$. But $\nabla g(ap) = \frac{1}{a}\nabla f(p)$

$\nabla g/\|\nabla g\|\big|_{ap} = \frac{1}{a}\nabla f(p) / \|\frac{1}{a}\nabla f(p)\| = \nabla f/\|\nabla f\| = N$. So N is also in Gauss Image of $\tilde{S}$

$\forall N$ in Gauss image of $\tilde{S}$, $\exists q$ s.t. $\nabla g(q)/\|\nabla g(q)\| = N$. But $\nabla g(q) = \frac{1}{a}\nabla f(q/a)$

$\nabla f(q/a)/\|\nabla f(q/a)\| = a\nabla g(q)/\|a\nabla g(q)\| = \nabla g(q)/\|\nabla g(q)\| = N$.

So the spherical images of S and $\tilde{S}$ are the same

(c) $\forall v \in S_p$, $k(v) = -\nabla_v N \cdot v$, $\nabla_v N = (\nabla N_1(p)\cdot v, \cdots, \nabla N_{n+1}(p)\cdot v)^T \cdot v$

$\nabla N_i(p) = \left(\frac{\partial N_i}{\partial x_1}(p), \cdots, \frac{\partial N_i}{\partial x_{n+1}}(p)\right)$. As short hand, denote $\nabla f = (f'_1, \cdots, f'_{n+1})$

So $\frac{\partial N_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{f'_i}{\|\nabla f\|} = \frac{1}{\|\nabla f\|^2}\left(f''_{ij}\|\nabla f\| - f'_i \cdot \frac{1}{\|\nabla f\|}\sum_{k=1}^n f'_k f''_{kj}\right) = \frac{1}{\|\nabla f\|^3}\left(f''_{ij}\|\nabla f\|^2 - f'_i \sum_{k=1}^n f'_k f''_{kj}\right)\Big|_p$ (1)

$\forall v \in \tilde{S}_{ap}$, $\tilde{k}(v) = -\nabla_v \tilde{N}\cdot v$. Using similar notation

$\frac{\partial \tilde{N}_i}{\partial x_j} = \frac{1}{\|\nabla g\|^3}\left(g''_{ij}\|\nabla g\|^2 - g'_i \sum_{k=1}^n g'_k f''_{kj}\right)\Big|_p$ (2) But $g(p) = f(p/a)$.

So $\nabla g(p) = \frac{1}{a}\nabla f(p/a)$, i.e. $\nabla g(ap) = \frac{1}{a}\nabla f(p)$, $g''_{ij}(p) = \frac{1}{a^2} f''_{ij}(p/a)$, i.e. $g''_{ij}(ap) = \frac{1}{a^2} f''_{ij}(ap)$

by plugging into (11,12)

So $\frac{\partial \tilde{N}_i}{\partial x_j}\Big|_{ap} = \frac{1}{a}\frac{\partial \tilde{N}_i}{\partial x_j}\Big|_p$ So $\tilde{k}(v) = k(v)/a$, which is true at all the (shared) stationary points. on $\|v\|=1$

But mean curvature H is the average of k at stationary points, thus $H(ap) = \frac{1}{a}H(p)$.

(d) K (Gauss-Kronecker curvature) is the product of $k(v)$ at stationary points

So $\tilde{K}(ap) = a^{-n}K(p)$

Remark Above argument based on stationary points is not strict enough, especially considering the multiplicity of L_p's eigenvalues. A better proof is: As $\nabla g(ap) = \frac{1}{a}\nabla f(p)$

So $S_p = \tilde{S}_{ap}$. $\forall v, w \in S_p$ ($\tilde{S}_{ap}$). $L_p(v)\cdot w = \frac{1}{2}[k(v+w) - k(v) - k(w)]$ as $\tilde{k}(\cdot) = k(\cdot)/a$

let $\tilde{L}_{ap}$ be Weingarten map on $\tilde{S}$ at ap $\rightarrow \tilde{L}_{ap}(v)\cdot w = \frac{1}{2}[\tilde{k}(v+w) - \tilde{k}(v) - \tilde{k}(w)] = \frac{1}{a}L_p(v)\cdot w$

Since w is arbitrary in S_p, so $\tilde{L}_{ap}(v) = \frac{1}{a}L_p(v)$ $\forall v$. So each eigenvalue λ_i of L_p corresponds to the eigenvalue λ_i/a of $\tilde{L}_{ap}$. As H and K are average / product of eigenvalues, we have

$H(ap) = \frac{1}{a}H(p)$ $\quad K(ap) = a^{-n}K(p)$

Proof: set $w = \tilde{L}_{ap}(v) - \frac{1}{a}L_p(v) \in S_p$

then one has $(\tilde{L}_{ap}(v) - \frac{1}{a}L_p(v))\cdot(\tilde{L}_{ap}(v) - \frac{1}{a}L_p(v)) = 0$. So $\tilde{L}_{ap}(v) = \frac{1}{a}L_p(v)$

13.1 If S is convex at p, then h_u ($u = N(p)$ Gauss map) attains local max/min at p. So $\mathcal{H}_p$ is semi-definite, so $\mathcal{L}_p = \pm\mathcal{H}_p$ is also semi-definite. But the Gauss-Kronecker curvature, which is equal to the product of all eigenvalues of $\mathcal{L}_p$, is negative. As S_p is even dimensional, there must be one positive and one negative eigenvalue, which contradicts the fact that $\mathcal{L}_p$ is semi-definite. So S is not convex at p.

13.2 $\forall v, w \in S_p$ $\quad \nabla_v(\text{grad } h)\, w = \nabla_v(\nabla h - (\nabla h\cdot N)N)\, w = \nabla_v(\nabla h)\, w - (\nabla h\cdot N)(\nabla_v N\cdot w)$

$\nabla_w(\text{grad } h)\, v = \nabla_w(\nabla h - (\nabla h\cdot N)N)\, v = \nabla_w(\nabla h)\, v - (\nabla h\cdot N)(\nabla_w N\cdot v)$

We know L_p is self-adjoint, i.e., $\nabla_v N\cdot w = \nabla_w N\cdot v$. Besides,

$\nabla_v(\nabla h)\, w = v^T H w = w^T H v = \nabla_w(\nabla h)\, w$ $\quad$ So $\nabla_v(\text{grad } h)\, w = \nabla_w(\text{grad } h)\, v$, so self-adjoint

13.3. (a) $\Rightarrow$ If $\mathcal{Q}$ is pos Def, then $\forall$ eigenvector v (value λ), $\mathcal{Q}(v) = \lambda v$, $\mathcal{Q}(v)\cdot v = \lambda > 0$ as $\mathcal{Q}$ is Pos Def

$\Leftarrow$ We know that the eigenvectors $v_1 \cdots v_n$ make up an orthonormal basis on S_p. $\forall v \in S_p$.

Let $v = \sum_{i=1}^n a_i v_i$ $\quad \mathcal{L}(v)\cdot v = \mathcal{Q}(\sum_{i=1}^n a_i v_i)\cdot \sum_{i=1}^n a_i v_i = (\sum_{i=1}^n a_i \mathcal{Q}(v_i))(\sum_{i=1}^n a_i v_i) = (\sum_{i=1}^n a_i \lambda_i v_i)(\sum_{i=1}^n a_i v_i)$

$= \sum_{i=1}^n a_i^2 \lambda_i \geq 0$ because $\lambda_i > 0$.

It is equal to 0 iff $a_i = 0$ i.e. $v = 0$

(b) $\Leftarrow$ Since L is self-adjoint linear transformation, its associated matrix L is symmetric so it has two real valued eigenvalues λ_1, λ_2. $\det L > 0 \Rightarrow \lambda_1\lambda_2 > 0$ But if $\lambda_1 < 0, \lambda_2 < 0$, then L is negative definition, i.e. there can't be any v: $\mathcal{Q}(v) > 0$.

thus $\lambda_1 > 0, \lambda_2 > 0$.

$\Leftarrow$ by definition $\mathcal{Q}(v) > 0$ for all $v \neq 0$. As $\mathcal{Q}$ is Pos Def. both eigenvalues are positive, thus $\det L = \lambda_1 \lambda_2 > 0$

(c) L is non-singular $\Leftrightarrow \det L = \prod_{i=1}^{n} \lambda_i \neq 0 \Leftrightarrow \lambda_i \neq 0$ (λ_i are eigenvalues)

$\mathcal{Q}$ is non-degenerate $\Leftrightarrow$

$\mathcal{H}_p$ is non-degenerate $\Leftrightarrow L: v \mapsto \nabla_v(\text{grad } h)$ is non-singular $\Leftrightarrow \lambda_i \neq 0 \Leftrightarrow \forall v$ $\nabla_v(\text{grad } h) \neq 0$ (i.e. p is non-degenerate)

13.4 If h is height function or any function which has constant $\|\nabla h\|$, then

$(h\circ\beta)'(t) = \nabla h(\beta(t)) \cdot \dot\beta(t) \leq \|\nabla h(\beta(t))\| \cdot \|\dot\beta(t)\| = \|\nabla h(\alpha(t))\| \cdot \|\dot\alpha(t)\|$

$= \nabla h(\alpha(t)) \cdot \dot\alpha(t) = (h\circ\alpha)'(t)$.

So $h(\alpha(b)) = h(\alpha(a)) + \int_a^b (h\circ\alpha)'(t) \geq h(\beta(a)) + \int_a^b (h\circ\beta)'(t)\,dt = h(\beta(b))$

Equality holds iff $\nabla h(\beta(t)) = \lambda \dot\beta(t)$ $\lambda \geq 0$. But $\nabla h(\alpha(t)) = \nabla h(\beta(t))$.

So $\|\nabla h(\alpha(t))\| = \lambda \|\dot\beta(t)\| = \lambda \|\dot\alpha(t)\|$ But $\dot\alpha(t) = \nabla h(\alpha(t))$. So $\lambda = 1$

So $\nabla h(\beta(t)) = \dot\beta(t)$, i.e. β is also a gradient line passing thru $\alpha(a)$, but such a line is unique, so $\beta = \alpha$.

If $\|\nabla h\| = \text{const}$ is not guaranteed, WE FEEL that this proposition may not hold. (this $\tilde h$ is actually the h in the question)

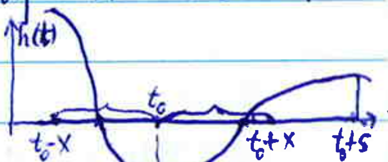
The following is a counter-example. Let $\tilde h(x_1, x_2) = h(x_1)$ $f(x_1, x_2) = x_2$

then $\nabla\tilde h = (h'(x_1), 0)$ $\nabla f = (0, 1)$ so $S = f^{-1}(0)$ is n-surface. $\nabla\tilde h \perp \nabla f \Rightarrow \text{grad } \tilde h = \nabla\tilde h$

$\alpha(t), \beta(t) \in S$. So we can write in brief $\alpha(t) = (\alpha(t), 0)$. $\beta(t) = (\beta(t), 0)$ (here $R \to R$) ($R \to R$)

So now $\dot\alpha(t) = h'(\alpha(t))$ $|\dot\beta(t)| = |\dot\alpha(t)|$

As $\beta(t)$ appears in the conclusion only inside $h(\beta(t))$, the only constraint on β is actually $\ell(\beta) = \int_a^b \|\dot\beta(t)\|dt = \int_a^b \|\dot\alpha(t)\|dt = \ell(\alpha)$. Now we check function $h(t) = \frac{1}{t}\sin\frac{1}{t}$ $(t>0)$ so $\tilde h$ and f are defined on (R^+, R) which is open. Let $\alpha(a) = t_0$, s.t. $\sin\frac{1}{t_0} = -1$



the first peak to the left of t_0 is $t_0 - X$, where $\frac{1}{t_0 - X} = \frac{1}{t_0} + \pi$, $X = \frac{\pi t_0^2}{1+\pi t_0}$.

$h(t_0 - X) = \frac{1}{t_0} + \pi$, $h(t_0+X) = \frac{1+\pi t_0}{t_0 + 2\pi t_0^2}\sin\frac{1+\pi t_0}{t_0+2\pi t_0^2} < \frac{1+\pi t_0}{t_0+2\pi t_0^2} < \frac{1+\pi t_0}{t_0} = h(t_0 - X)$.

Besides, the first peak to the right of t_0 is $t_0 + S$, s.t. $\frac{1}{t_0+S} = \frac{1}{t_0} - \pi$, $S = \frac{\pi t_0^2}{1-\pi t_0} > X$

in (t_0, t_0+S) $\dot\alpha > 0$. Now suppose $\alpha(a) = t_0 + \varepsilon$ where $\varepsilon > 0$ is sufficiently small,

. for $t > a$, $\alpha(t)$ monotonically increases, and $\beta(t)$ is forced to decrease monotonically.

As ε can be arbitrarily small, by above discussion, $\beta(t)$ first reaches $t_0 - X$, while $\alpha(t)$ hasn't reached $t_0 + S$, i.e. $\dot\alpha = h\circ\alpha > h'(t_0+\varepsilon)$ guarantees that α has enough impetus to go right and meanwhile, β goes left. β reaches $t_0 - X$ while α only reaches $t_0 + X + 2\varepsilon$.

Suppose b is chosen (stop) at such a moment, then we have $h(\alpha(b)) < h(\beta(b))$ which contradicts the exercise assertion.

13.5 $h(\beta(t)) = c \Rightarrow \nabla h(\beta(t_1)) \cdot \dot{\beta}(t_1) = 0$, $\alpha(t_0) = \beta(t_1)$ $\Rightarrow$ $\nabla h(\alpha(t_0)) \cdot \dot{\beta}(t_1) = 0$, $\dot{\alpha}(t_0) = (\text{grad } h)(\alpha(t_0))$

~~$\Rightarrow (\text{grad } h)(\alpha(t_0)) \cdot \dot{\beta}(t_1) = 0 \Rightarrow \nabla h$~~

$\Rightarrow \dot{\alpha}(t_0) \cdot \dot{\beta}(t_1) = (\text{grad } h)(\alpha(t_0)) \cdot \dot{\beta}(t_1) = \left(\nabla h(\alpha(t_0)) - (\nabla h(\alpha(t_0)) \cdot N(\alpha(t_0))) N(\alpha(t_0)) \right) \cdot \dot{\beta}(t_1) = 0$

$= \nabla h(\alpha(t_0)) \cdot \dot{\beta}(t_1) = 0$ As $N(\alpha(t_0)) \cdot \dot{\beta}(t_1) = N(\beta(t_1)) \cdot \dot{\beta}(t_1) = 0$.