Exercises in

Elementary Topics in Differential Geometry by J.A. Thorpe

1.10 $\text{graph}(f)=\{(x_1\cdots x_{n+1})\in R^{n+1} : (x_1\cdots x_n)\in U,\ x_{n+1}=f(x_1\cdots x_n)\}$
Then graph(f) is a level set for $F(x_1\cdots x_{n+1})=0$, where $F(x_1\cdots x_{n+1})=f(x_1\cdots x_n)-x_{n+1}$

2.4 As integral curve, $\dot\alpha(t)=X(\alpha(t))$. If it crosses itself, then there exists t_1, t_2.
s.t. $\alpha(t_1)=\alpha(t_2)$, $\dot\alpha(t_1)\neq\dot\alpha(t_2)$. But that isn't allowed.

2.7 (a) complete (b) incomplete say $p=(-1,0)$ (c) complete
(d) $x_1=tg(t+c)$, so $t\neq -c+\frac{\pi}{2}$. incomplete

2.8 Define $\tilde\beta(t)=\beta(t+t_0)$ $t\in\{x-t_0|x\in\tilde I\}$ then $\tilde\beta(0)=P$, $\dot{\tilde\beta}(t)=\dot\beta(t+t_0)=X(\beta(t+t_0))=X(\tilde\beta(t))$ $(t\in\tilde I-t_0)$ offset
So $\tilde\beta(t)$ is an integral curve of X with $\tilde\beta(0)=P$. Since $\alpha(t)$ is the maximal of such curves, So for $\forall t\in\{x-t_0|x\in\tilde I\}$. $\tilde\beta(t)=\alpha(t)$ i.e. $\beta(t)=\alpha(t-t_0)$ $\forall t\in\tilde I$

2.9 Define $\beta(t)\triangleq\alpha(t)$ $t\in I$, so $\beta(t_0)=\alpha(0)$. β is an integral curve of X on I
By Ex. 2.8. $\beta(t)=\alpha(t-t_0)$ i.e. $\alpha(t)=\alpha(t-t_0)$. i.e. α periodic.
(Don't worry about def. domain too much, only check & restrict in the last step)

2.10 (a) $\varphi_t(P)=P+(t,0)$ translation. obviously one-to-one $R^2\to R^2$
(b) $\varphi_0(P)=P+(0,0)=P$. $\varphi_{t_1+t_2}(P)=P+(t_1+t_2,0)=(P+(t_2,0))+(t_1,0)$
$\varphi_{-t}(P)=P+(-t,0)$ $\varphi_t(\varphi_{-t}(P))=P+(-t,0)+(t,0)=P$

2.11. (a) $\varphi_t(x_1,x_2)=(x_1\cos t-x_2\sin t,\ x_1\sin t+x_2\cos t)=\begin{pmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$
rotation by t. one-to-one, additive group obviously
(b) $\varphi_t(x_1,x_2)=(x_1e^t, x_2e^t)=(x_1,x_2)\cdot e^t$ scaling bijection $e^{t_1+t_2}=e^{t_1}\cdot e^{t_2}$ so additive
(c). $\varphi_t(x_1,x_2)=\frac12\begin{pmatrix}e^t+e^{-t} & e^t-e^{-t}\\ e^t-e^{-t} & e^t+e^{-t}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$, det $=\frac12\cdot 4$, invertible
$=\frac12(e^t+e^{-t})\begin{pmatrix}1 & \tanh(t)\\ \tanh(t) & 1\end{pmatrix}$ Use $\tanh(t_1+t_2)=\frac{\tanh(t_1)+\tanh(t_2)}{1-\tanh(t_1)\cdot\tanh(t_2)}$

2.12 Suppose $\beta(t)$ is the integral curve of X with $\beta(0)=\varphi_{t_2}(P)$. so $\alpha(0)=P$. $\alpha(t_2)=\beta(0)$

By using Ex. 2.8 (now the β here is the α is Ex. 2.8), $\beta(t)=\alpha(t+t_2)$
$\varphi_{t_1}(\varphi_{t_2}(P))=\beta(t_1)=\alpha(t_1+t_2)=\varphi_{t_1+t_2}(P)$, $\varphi_{t_2}(\varphi_{-t_2}(P))=\beta(-t_2)=\alpha(0)=P$ $\forall t_2$