(b). By definition, we should have: $X\cdot\dot\alpha=0$, $X\cdot N\circ\alpha=0$ and $X^*=0$

$X^*=0 \Rightarrow \dot X-(\dot X\cdot N\circ\alpha)N\circ\alpha-(\dot X\cdot\dot\alpha)\dot\alpha+(\dot X\cdot N\circ\alpha)(N\circ\alpha\cdot\dot\alpha)\dot\alpha=0$ (*) Note: $\dot\alpha\perp N\circ\alpha$

$X\cdot\dot\alpha=0\Rightarrow \dot X\dot\alpha+X\cdot\ddot\alpha=0$, $X\cdot N\circ\alpha=0\Rightarrow \dot X\cdot N\circ\alpha+X\cdot N\dot\circ\alpha=0$

Plugging into (*): $\dot X+(X\cdot N\dot\circ\alpha)N\circ\alpha+(X\cdot\ddot\alpha)\dot\alpha=0$. 1st order differential equation together with initial condition $X(t_0)=V$. So there exists a unique solution $X(t)$.

Now check $X\cdot\dot\alpha=0$ and $X\cdot N\circ\alpha=0$

$(X\cdot\dot\alpha)^{\circ}=\dot X\dot\alpha+X\ddot\alpha=X\ddot\alpha-(X\cdot N\dot\circ\alpha)(N\circ\alpha\cdot\dot\alpha)-(X\cdot\ddot\alpha)(\dot\alpha\cdot\dot\alpha)=0$

$(X\cdot N\circ\alpha)^{\circ}=\dot X\cdot N\circ\alpha+X\cdot(N\dot\circ\alpha)=X\cdot(N\dot\circ\alpha)-(X\cdot N\dot\circ\alpha)(N\circ\alpha\cdot N\circ\alpha)-(X\cdot\ddot\alpha)(\dot\alpha\cdot N\circ\alpha)=0$

domain check same as Thm 1 in chapter as $\|X\|$ is constant

(c) (i) F_α is linear map, If V and W are Fermi parallel along α, then so are V+W and cV ($c\in\mathbb{R}$)

(ii) F_α is one to one and onto: the kernel of F_α is zero because $\|F_\alpha(V)\|=0 \Leftrightarrow V=0$ by (iii).
so F_α is one-to-one from one n-dim vector space to another. But such maps are onto

(iii) $F_\alpha(V)\cdot F_\alpha(W)=V\cdot W$, $\forall V,W\in\dot\alpha(a)^\perp$ because $(X\cdot Y)^{\circ}=X^*Y+XY^*=0$, i.e. $X\cdot Y$ is constant

9.1 (a) $\nabla f=(4x_1,6x_2)$ $\quad \nabla_v f(p)=(4,0)\cdot(2,1)=8$

(b) $\nabla f=(2x_1,-2x_2)$ $\quad \nabla_v f(p)=(2,-2)\cdot(\cos\theta,\sin\theta)=2(\cos\theta-\sin\theta)$

(c) $\nabla f=(x_2x_3^2,\ x_1x_3^2,\ 2x_1x_2x_3)$, $\nabla_v f(p)=(1,1,2)\cdot(a,b,c)=a+b+2c$

(d) $\nabla f=(q,2q)$ $\quad \nabla_v f(p)=2p\cdot v$

9.2 $\nabla_{e_i}f=\left(\frac{\partial f(p)}{\partial x_1},\cdots\frac{\partial f(p)}{\partial x_{n+1}}\right)\cdot(0,\cdots1,\cdots0)=\frac{\partial f(p)}{\partial x_i}$

9.3 (a) $\nabla X_1=(x_2,x_1)$ $\quad \nabla X_2=(0,2x_2)$ $\quad \nabla_v X=((0,1)\cdot(0,1),(0,0)\cdot(0,1))=(1,0)$

(b) $\nabla X_1=(0,-1)$ $\quad \nabla X_2=(1,0)$ $\quad \nabla_v X=((0,-1)(-\sin\theta,\cos\theta),(1,0)(-\sin\theta,\cos\theta))=(-\cos\theta,-\sin\theta)$

(c) $\nabla X_i=(q,2e_i)$ $\quad \nabla_v X=(2,2,\cdots2)$.

9.5 Let $Y(t)=X(\alpha(t))$ be the vector field tangent to S along α. As $D_vX=(X\circ\alpha)'(t_0)$ $\quad F(t)=f(\alpha(t))$ $\quad D_vf=F'(t_0)$
where $\alpha: I\to S$ is any parametrized curve in S with $\dot\alpha(t_0)=v$. Then quote the properties i-iii in chapter 8 on page 46. Note in (iii) $\nabla_v(X\cdot Y)$ rather than $D_v(X\cdot Y)$ ($\nabla_v X\cdot Y=(X\cdot Y)'$)

9.4 Same as 9.5. $\nabla_v X=(X\circ\alpha)(t_0)$ $\quad \nabla_v f=\dot F$. Then quote the properties i-iii in Chapter 7 on Pg 39

9.6. $X(q)\cdot X(q)=1$ By property iii of ch 9 on Pg 54. $\nabla_v X(q)\cdot X(q)=\nabla_v 1=0$ i.e. $\nabla_v X\perp X(p)$