If X is tangent to S, then $\nabla_v X \cdot N(p) = (\dot{X \circ \alpha})(t_0) \cdot N(p) = 0$. So $\nabla_v X = D_v X$. So $D_v X \perp X(p)$

by proof in Thm 1 of chapter 5, then α is ~~unique~~ on S

9.7 (a) "if part": $\forall$ ~~parametric~~ integral curve $\alpha: I \to S$, s.t. $\alpha(t_0) = p$. $\dot\alpha(t) = X(\alpha(t))$, ~~then the geodesic is~~

$D_{X(p)} X = 0 \Leftrightarrow \nabla_{X(p)} X \parallel N(p) \Leftrightarrow (\dot{X \circ \alpha}) \parallel N \circ \alpha$

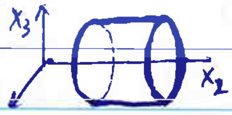
$\dot\alpha(t) = X \circ \alpha \Rightarrow \ddot\alpha = (\dot{X \circ \alpha})$

$\Rightarrow \ddot\alpha \parallel N \circ \alpha \Rightarrow \alpha$ is geodesic.

"only if" part: $\forall p \in S$, construct integral curve $\alpha: I \to$, s.t. $\alpha(t_0) = p$, $\dot\alpha(t) = X(\alpha(t))$

as X is tangent to S, α must be on S by proof in Thm 1 in chapter 5. By assumption $\ddot\alpha \in S_\alpha^\perp$ (geodesic)

As $\dot\alpha(t) = X(\alpha(t))$, we have $\ddot\alpha = (\dot{X \circ \alpha}) \in S_\alpha^\perp$ i.e. $(\dot{X \circ \alpha}) \parallel N \circ \alpha \Leftrightarrow D_{X(p)} X = 0$

(b)  cylinder $X(p) = (p, (0,1,0))$

9.8 (a) $N = (a_1, \dots, a_{n+1})$ $\nabla N_i = 0$ $L_p(v) = 0$

(b) $N = (0, \frac{X_2}{a}, \frac{X_3}{a})$. $\nabla N_1 = (0,0,0)$, $\nabla N_2 = (0, \frac{1}{a}, 0)$, $\nabla N_3 = (0, 0, \frac{1}{a})$, $L_p(v) = -(0, \frac{v_2}{a}, \frac{v_3}{a})$ (let $a > 0$)

9.9 By property (ii) on page 54. $\nabla_v(-N) = \nabla_v(-1) \cdot (N) + (-1) \nabla_v(N) = -\nabla_v N$

9.10 (a) suppose $L^*(e_i) = \sum_{i=1}^n \lambda_i e_i$, then by $L^*(e_i) e_j = e_j L(e_i)$ we have $\lambda_j = e_j \cdot L(e_i)$

So $L^*(e_i) = \sum_{j=1}^n (e_i \cdot L(e_j)) e_j$, $\forall v = \sum_{i=1}^n \alpha_i e_i \in V$, $L^*(v) = \sum_{i=1}^n \alpha_i \sum_{j=1}^n (e_i \cdot L(e_j)) \cdot e_j$. Let $w = \sum_{i=1}^n \beta_i e_i$

$L^*(v) \cdot w = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \alpha_i \beta_k (e_i \cdot L(e_j)) \cdot (e_j \cdot e_k) = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \beta_j (e_i \cdot L(e_j))$

$v \cdot L(w) = \sum_{i,j=1}^n \beta_i \alpha_j L(e_i) \cdot e_j = \sum \alpha_i \beta_j (e_i \cdot L(e_j)) = L^*(v) \cdot w$

So the only possible choice of $L^*(e_i) = \sum_{j=1}^n (e_i \cdot L(e_j)) e_j$ satisfies $v \cdot L(w) = L^*(v) \cdot w$ $\forall v, w \in V$.

(b) If ~~$L^* = L$~~ $L(v) = Av$ $\forall v, w \in V$. For $\forall w \in V$ $w \cdot L(v) = wAv$,

If we ~~set~~ choose $L^*(v) = A^T v$. then $v \cdot L^*(w) = v \cdot A^T w = wAv = wL(v)$.

As (a) proves L^* is unique and each linear transform corresponds to a unique matrix we know L^* correspond to A'. So $L^* = L \Leftrightarrow A$ is symmetric. So L_p is symmetric by Thm 2 (Pg 56)

9.11 $\forall i \in \{1, \dots, n\}$. $L_p(e_i) = -\nabla_{e_i} N(p) = -((\nabla N_1(p) \cdot e_i), \dots, (\nabla N_{n+1}(p) \cdot e_i))$

$\forall j \in \{1, \dots, n\}$. $\nabla N_j(p) e_i = \frac{\partial N_j}{\partial x_i}\Big|_p = \frac{\partial}{\partial x_i}\left(\frac{1}{\|\nabla f\|} \frac{\partial f}{\partial x_j}\right)\Big|_p = \frac{\partial}{\partial x_i}\left(\frac{1}{\|\nabla f\|}\right)\Big|_p \cdot \frac{\partial f}{\partial x_j}\Big|_p + \frac{1}{\|\nabla f\|} \frac{\partial^2 f}{\partial x_i \partial x_j}$

Since S is n-surface $\left|\frac{\partial}{\partial x_i}\left(\frac{1}{\|\nabla f\|}\right)\right|_p < \infty$. But $\nabla f(p) / \|\nabla f(p)\| = e_{n+1} \Rightarrow \frac{\partial f}{\partial x_j}\Big|_p = 0$ $\forall j \in \{1 \dots n\}$

So $\nabla N_j(p) e_i = \frac{1}{\|\nabla f\|} \frac{\partial^2 f}{\partial x_i \partial x_j}$

Since $N(p) = e_{n+1}$, and L_p is map $S_p \mapsto S_p$. So $\nabla N_{n+1}(p) \cdot e_i = 0$, thus

$$L_p(e_i) = -\sum_{j=1}^n \frac{1}{\|\nabla f\|} \frac{\partial^2 f}{\partial x_i \partial x_j} e_j$$

By the way, we can prove that $\nabla N_{n+1}(p) \cdot e_i = 0$. First $\frac{\partial f}{\partial x_{n+1}}\Big|_p = \|\nabla f(p)\|$