Second. $\frac{\partial}{\partial x_i}\frac{1}{\|\nabla f\|}\Big|_p = \frac{\partial}{\partial x_i}\left[\sum_{k=1}^{n+1}\left(\frac{\partial f}{\partial x_k}\right)^2\right]^{-\frac{1}{2}}\Big|_p = \left[\sum_{k=1}^{n+1}\left(\frac{\partial f}{\partial x_k}\right)^2\right]^{-\frac{3}{2}}\cdot\frac{-1}{2}\cdot\sum_{k=1}^{n+1} 2\frac{\partial f}{\partial x_k}\frac{\partial^2 f}{\partial x_k\partial x_i}\Big|_p$. But $\frac{\partial f}{\partial x_k}\Big|_p = \begin{cases} 0 & k=1,\cdots,n \\ \|\nabla f\| & k=n+1\end{cases}$

$= -\|\nabla f\|^{-3}\cdot\|\nabla f\|\frac{\partial^2 f}{\partial x_{n+1}\partial x_i} = -\|\nabla f\|^{-2}\frac{\partial^2 f}{\partial x_{n+1}\partial x_i}$ So L_p is Symmetric

So $\nabla N_{n+1}(p)\cdot e_i = -\|\nabla f\|^{-2}\frac{\partial^2 f}{\partial x_{n+1}\partial x_i}\|\nabla f\| + \|\nabla f\|^{-1}\frac{\partial^2 f}{\partial x_{n+1}\partial x_i} = 0$.

9.12 (a) Suppose a parametrized curve $\alpha: I\to S$. $\alpha(t_0)=p$. $\dot\alpha(t_0)=X(\alpha(t_0))$

$\nabla_{X(p)}Y = \dot{Y}\circ\alpha$, $\nabla_{X(p)}Y\cdot N(p) = \dot{Y}\circ\alpha\cdot N\circ\alpha$.

But as Y is tangent to S. $(Y\circ\alpha)\cdot(N\circ\alpha)=0 \Rightarrow (\dot{Y}\circ\alpha)(N\circ\alpha)+(Y\circ\alpha)(\dot{N}\circ\alpha)=0$

So $\nabla_{X(p)}Y\cdot N(p) = -(Y\circ\alpha)\cdot(\dot{N}\circ\alpha)\big|_{t_0} = -Y(p)\cdot L_p(X(p))$

Similarly, one can prove $\nabla_{Y(p)}X\cdot N(p) = -X(p)\cdot L_p(Y(p))$

By Thm 2 (Pg 55) $L_p(X(p))\cdot Y(p) = X(p)\cdot L_p(Y(p))$ Thus $\nabla_{X(p)}Y\cdot N(p) = \nabla_{Y(p)}X\cdot N(p)$

(b) by (a) obvious

9.13. For $\forall v$, define a parametrized curve $\alpha: I\to U$, $\alpha(t_0)=p$. $\dot\alpha(t_0)=v$. For $\forall\varepsilon$ there exists a δ s.t. $\|X(p+v)-X(p)-X'(p)(v)\|/\|v\|<\varepsilon$, $\forall \|v\|<\delta$. As α is continuous, there exists $\delta_1>0$ s.t. $\forall t\in(t_0-\delta_1, t_0+\delta_1)$: $\|\alpha(t)-\alpha(t_0)\|<\delta$. thus

$\|X(p+\alpha(t)-\alpha(t_0)) - X(p) - X'(p)(\alpha(t)-\alpha(t_0))\| / \|\alpha(t)-\alpha(t_0)\| < \varepsilon$

i.e. $\|X(\alpha(t)) - X(\alpha(t_0)) - X'(p)(\alpha(t)-\alpha(t_0))\| / \|\alpha(t)-\alpha(t_0)\| < \varepsilon$.

i.e. $\lim_{t\to t_0}\|X(\alpha(t)) - X(\alpha(t_0)) - X'(p)(\alpha(t)-\alpha(t_0))\| / \|\alpha(t)-\alpha(t_0)\| = 0$ (*)

Notice $\lim_{t\to t_0}\frac{\alpha(t)-\alpha(t_0)}{t-t_0} = \dot\alpha(t) = v$ So $\lim_{t\to t_0}\frac{\|\alpha(t)-\alpha(t_0)\|}{|t-t_0|} = \|v\|$ (1)

$\lim_{t\to t_0}(X(\alpha(t))-X(\alpha(t_0)))/(t-t_0) = \nabla_v X$ (by definition of $\nabla_v X$) (2)

As $X'(p)$ is a linear map, ~~suppose its corresponding matrix is A~~, thus

$\lim_{t\to t_0}\frac{X'(p)(\alpha(t)-\alpha(t_0))}{t-t_0} = \lim_{t\to t_0} X'(p)\left(\frac{\alpha(t)-\alpha(t_0)}{t-t_0}\right)$ (since $X'(p)$ is linear)

$= X'(p)\left(\lim_{t\to t_0}\frac{\alpha(t)-\alpha(t_0)}{t-t_0}\right) = X'(p)(v)$ (3)

If $\lim_{i\to\infty} v_i = v$ then $\lim_{i\to\infty} A(v_i) = A(v)$ use basis expression must finite dimensional

Plugging (1) (2) (3) into (*) $\|\nabla_v X - X'(p)(v)\|/\|v\| = 0$ i.e. $\nabla_v X = X'(p)(v)$

9.14 $L_p(p,v) \overset{\text{def of } L_p}{=} -\nabla_v N(p) \overset{\text{def. of } \tilde N}{=} -\nabla_v \tilde N(p) \overset{\text{result of 9.13}}{=} -\tilde N'(p)(v)$

9.15 (a) $\dot{\tilde\alpha}(t) = X(\tilde\alpha(t)) \Leftrightarrow \begin{cases}\dot x_1 = u_1, \cdots, \dot x_{n+1} = u_{n+1} \\ \dot u_k = -(u_1,\cdots,u_{n+1})\cdot\left(\sum_{i=1}^{n+1}u_i\frac{\partial N_1}{\partial x_i},\cdots,\sum_{i=1}^{n+1}u_i\frac{\partial N_{n+1}}{\partial x_i}\right)N_k = -N_k\sum_{i,j=1}^{n+1}u_i u_j\frac{\partial N_j}{\partial u_i}\end{cases}$ (denote $\alpha=(x_1,\cdots,x_{n+1})$, $\tilde\alpha=(\alpha,u_1,\cdots,u_{n+1})$)

So $\ddot x_k + N_k\cdot\sum_{i,j=1}^{n+1}\dot x_i\dot x_j\frac{\partial N_j}{\partial x_i} = 0$ which is the same as (G) f

Then follow the proof in the theorem of chapter 7, α is a geodesic of S. (Note: $\alpha\subset S$ is assumed)

Note the equation $\dot{\tilde\alpha}(t) = X(\tilde\alpha(t))$ is 1st order differential system in u and x so unique solution

(b) $X(\beta(t)) = \dot\beta(t) \Leftrightarrow (\dot\beta_1 = \beta_2$ and $\dot\beta_2 = -(\beta_2\cdot\nabla_{\beta_2}N)N(\beta_1)$ As in (a), we can