3.1 $n=1$ $f = x_1^2 - x_2^2$ $f^{-1}(-1)$ is $x_1^2 = x_2^2+1$ $\nabla f = (2x_1, -2x_2)$ so $\nabla f \neq 0$, no such $p$

$f^{-1}(1)$ also doesn't have such $p$. $f^{-1}(0)$, $\nabla f(0,0) = (0,0)$, $f^{-1}(0)$ is $x_1 = \pm x_2$

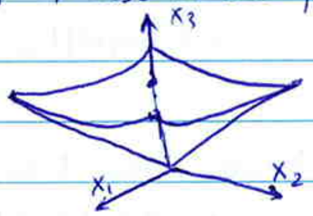
Its tangent space is $\{\lambda(1,1), \lambda(1,-1) \mid \lambda \in R\} \neq [\nabla f(0,0)]^{\perp} = R^2$

$n=2$ $f = x_1^2 + x_2^2 - x_3^2$ $f^{-1}(-1)$ is $x_1^2 + x_2^2 = x_3^2 + 1$. $\nabla f \neq 0$ no such $p$. $f^{-1}(1)$ also no such $p$

$f^{-1}(0)$: $x_3^2 = x_1^2 + x_2^2$ at $p = (0,0)$, the tangent space

at $(0,0,0)$ is all vectors $\overset{(x_1,x_2,x_3)}{V}$ where $V$ is $45°$ to $x_3$ axis

$|V \cdot (0,0,1) / \|V\|| = \frac{\sqrt{2}}{2}$ i.e. $x_3^2 = x_1^2 + x_2^2 \neq [\nabla f(0)]^{\perp}$




3.2 (a) the example in 3.1 with $n=1$, $c=0$. $f^{-1}(0)$: $x_1 = \pm x_2$ $(1,1), (1,-1) \in S$, $(1,0) \notin S$

(b) $f(x_1 \cdots x_{n+1}) = c$. $S = f^{-1}(c)$, tangent space $= R_p^{n+1}$

3.4 $f \circ \alpha = c \Leftrightarrow \frac{d(f(\alpha(t)))}{dt} = 0 \Leftrightarrow \nabla f(\alpha(t)) \cdot \dot\alpha(t) = 0 \Leftrightarrow \dot\alpha \perp \nabla f(\alpha)$ $\forall t$.

3.5 $\alpha$ is integral curve of $\nabla f \Rightarrow \dot\alpha = \nabla f(\alpha)$

(a) $\frac{d}{dt} f(\alpha(t)) = \nabla f(\alpha(t)) \cdot \dot\alpha(t) = \|\nabla f(\alpha(t))\|^2$

(b) $\frac{d}{dt} f(\beta(s_0)) = \nabla f(\beta(s_0)) \cdot \dot\beta(s_0) = \nabla f(\alpha(t_0)) \cdot \dot\beta(s_0)$. As $\|\dot\beta(s_0)\| = \|\dot\alpha(t_0)\|$

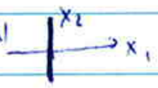
it is maximized when $\dot\beta(s_0) = \dot\alpha(t_0) = \nabla f(\alpha(t_0))$, then

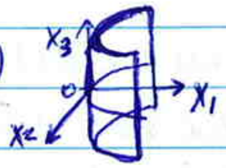
$\frac{d}{dt} f(\beta(s_0)) = \|\nabla f(\alpha(t_0))\|^2 = \frac{d}{dt} f(\alpha(t_0))$ by (a)

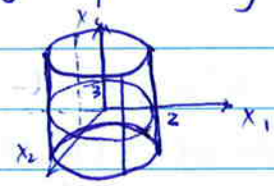
4.3 Consider $S = f^{-1}(c)$. $\forall p \in S$, $p$ is an extreme point of $g$ on $S$.

By Lagrange Theorem, $\nabla g(p) = \lambda \cdot \nabla f(p)$ $\forall p \in S$. $\lambda \neq 0$ because $\nabla g(p) \neq 0$ for all $p \in S$

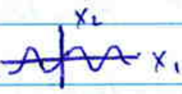
4.4 See http://users.rsise.anu.edu.au/~xzhang/dg_thorpe/monkey.jpg

4.5 (a) 

the $x_2$-axis (b) 

(c) 

ellipse on $x_3 = 0$

4.6 


4.7 $g(x_1, x_2, x_3) = f(x_1, (x_2^2 + x_3^2)^{1/2})$, then $\frac{\partial g}{\partial x_1} = \frac{\partial f}{\partial x_1}$. Denote $u = (x_2^2 + x_3^2)^{1/2}$

then $\frac{\partial g}{\partial x_2} = \frac{\partial f}{\partial u} x_2 (x_2^2 + x_3^2)^{-1/2}$, $\frac{\partial g}{\partial x_3} = \frac{\partial f}{\partial u} x_3 (x_2^2 + x_3^2)^{-1/2}$. If $\nabla g(p) = 0$, then

$\frac{\partial g}{\partial x_2} = \frac{\partial g}{\partial x_3} = 0$, i.e. $0 = (\frac{\partial g}{\partial x_2})^2 + (\frac{\partial g}{\partial x_3})^2 = [\frac{\partial f}{\partial u}]^2 = 0$. So $\frac{\partial f}{\partial u} = 0$. Besides $\frac{\partial f}{\partial x_1} = \frac{\partial g}{\partial x_1} = 0$

So $\nabla f = 0$ at $p$, which contradicts with the fact that $\nabla f$ ~~is a surface~~ $\neq 0$