So $\hat{\varphi}_u(t)\cdot\hat{\varphi}_x(t) = \cos(\theta_u(t)-\theta_x(t))$. By (*) $\cos(\theta_u(t)-\theta_x(t)) > 1-\varepsilon_2$ (**)

Let $\theta_0 = \arccos(1-\varepsilon_2)$. ~~As $\theta_u(a), \theta_x(a) \in [0,2\pi)$, $|\theta_u(a)-\theta_x(a)| < \theta_0$~~

So for $\forall t$, $\exists k_t \in \mathbb{Z}$, s.t. $|\theta_u(t)-\theta_x(t)| \in (2k_t\pi-\theta_0, 2k_t\pi+\theta_0)$

If $\exists t_1, t_2$, $t_1 < t_2$, $k_{t_1} \neq k_{t_2}$, $|\theta_u(t_1)-\theta_x(t_1)| \in (2k_{t_1}\pi-\theta_0, 2k_{t_1}\pi+\theta_0)$

$|\theta_u(t_2)-\theta_x(t_2)| \in (2k_{t_2}\pi-\theta_0, 2k_{t_2}\pi+\theta_0)$ Note $\theta_0 \in (0, \pi/2]$

As $\theta_u(t)-\theta_x(t)$ is continuous with respect to t, there must exist $t_3 \in (t_1, t_2)$

s.t. $\theta_u(t_3)-\theta_x(t_3) = 2\pi\cdot\min(k_{t_1}, k_{t_2}) + \pi$, ~~if ε_2 is small enough and thus θ_0 is small enough~~. But $\cos(\theta_u(t_3)-\theta_x(t_3)) = -1$ violating (**). Thus there is a $k \in \mathbb{Z}$,

s.t. $\forall t \in [a,b]$, $|\theta_u(t)-\theta_x(t)| \in (2k\pi-\theta_0, 2k\pi+\theta_0)$

But $|k(\varphi_u)-k(\varphi_x)| = \frac{1}{2\pi}|(\theta_u(b)-\theta_u(a))-(\theta_x(b)-\theta_x(a))|$

~~$\leq \frac{1}{2\pi}(|\theta_u(b)-\theta_x(b)| + |\theta_u(a)-\theta_x(a)|)$~~

$= \frac{1}{2\pi}|(\theta_u(b)-\theta_x(b)) - (\theta_u(a)-\theta_x(a))|$

$< \frac{1}{2\pi}2\theta_0 = \frac{\theta_0}{\pi}$ ~~$\pi/3$ is to make θ_0 sufficiently small to ensure $\exists t_3$~~

So $\forall u \in [a,b]$, $\forall \varepsilon \in (0, 1/2)$, $\exists \varepsilon_2 = 1-\cos(\varepsilon\pi)$, $\varepsilon_1 = \sqrt{2\varepsilon_2}$, s.t. $\forall x \in (u-\varepsilon_1, u+\varepsilon_1) \cap [0,1]$,

$|k(\varphi_x)-k(\varphi_u)| < \varepsilon$. So $k(\varphi_u)$ is a continuous function of u

Finally, as $k(\varphi_u)$ can only assume integer value, $k(\varphi_0) = k(\varphi_1)$

Note: φ can be on any $[a,b]$ ($a, b \neq \infty$) and $k(\varphi_a) = k(\varphi_b)$

11.18 (a) $\forall n$, define $\alpha(t) = (\cos nt, \sin nt)$, i.e. $\alpha'(t) = (\frac{1}{n}\sin nt, \frac{-1}{n}\cos nt)$

Then following Example 2 on Pg 75, $\int_{\dot\alpha}\eta = n\int_0^{2\pi}\frac{1}{\cos^2 t+\sin^2 t}dt = 2n\pi$

i.e. the rotation index of α is n.

(b) We follow the definitions of φ, ψ, ϕ as in the hint, but define to more formally

Let $u \in \mathbb{R}^2$, $u \neq 0$ define $h(t) = \alpha(t)\cdot u$. Since α is compact, h must attain its minimum, say, at t_0. ~~By Lagrange multiplier Thus, $u = \lambda$~~

So $h'(t_0) = \dot\alpha(t_0)\cdot u = 0$, i.e. $\dot\alpha(t_0) \perp u$. By definition, ϕ is continuous.

$k(\varphi_0)$ is the rotation index of α, because $\varphi_0(t) = \psi(t,t) = \dot\alpha(t)/\|\dot\alpha(t)\|$.

As for $k(\varphi_1)$, when $t \in (t_0, t_0+\tau/2]$ $\varphi_1(t) = \psi(t_0, -t_0+2t) = (\alpha(-t_0+2t)-\alpha(t_0))/\|\alpha(-t_0+2t)-\alpha(t_0)\|$ ($t_0 \to t_0+\tau$)

Now the η is exact because $\varphi_1(t)\cdot u \geq 0$ and We can set $v = -u$, and have ~~$\theta_v = \theta_u$~~

$V = \mathbb{R}^2 - \{rv \mid r>0\}$, η is exact on V. Here $\varphi_1(t)\cdot u \geq 0$, because $\forall t$

$h(t) \geq h(t_0) \Rightarrow (\alpha(t)-\alpha(t_0))\cdot u \geq 0 \Rightarrow \varphi_1(t)\cdot u \geq 0$ ($t \in (t_0, t_0+\tau/2)$)

So $\int_{\varphi_1|(t_0,t_0+\tau/2]}\eta = \theta_v(\varphi_1(t_0+\frac{\tau}{2})) - \theta_v(\varphi_1(t_0)) = \theta_v(-\dot\alpha(t_0)/\|\dot\alpha(t_0)\|) - \theta_v(\dot\alpha(t_0)/\|\dot\alpha(t_0)\|)$

$= \pm\pi$

For $t \in (t_0+\frac{\tau}{2}, t_0+\tau]$, $\varphi_1(t) = \psi(2t-t_0-\tau, t_0+\tau) = (\alpha(t_0)-\alpha(2t-t_0-\tau))/\|\cdot\|$ ($t_0 \to t_0+\tau$)