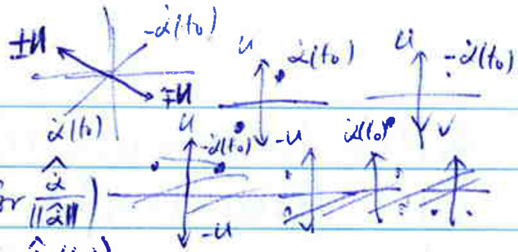

Now we set $V'=u$, $V'=\mathbb{R}^2-\{rv \mid r>0\}$. η is exact on V'

$\int_{\varphi_1|(t_0+T/2,\, t_0+T]} \eta = \theta_{v'}(\hat{\dot\alpha}(t_0)) - \theta_{v'}(-\hat{\dot\alpha}(t_0))$ ($\hat{\dot\alpha}$ stands for $\frac{\dot\alpha}{\|\dot\alpha\|}$)

But for $\forall u$, $\theta_u(-\hat{\dot\alpha}(t_0)) - \theta_u(\hat{\dot\alpha}(t_0)) + \theta_{-u}(\hat{\dot\alpha}(t_0)) - \theta_{-u}(-\hat{\dot\alpha}(t_0))$

$= \pm 2\pi$ (if $+u$, $-u$, then 2π, if $-u$, $+u$, then -2π)

Thus. ~~φ_0~~ $k(\varphi_0)=k(\varphi_1)=\pm 1$ i.e. rotation index is ± 1.

11.19 (a) $h'(t_0)=0 \iff \dot\alpha(t_0)\cdot u=0 \iff \dot\alpha(t_0)\perp u$ $\Big\}$ $\iff$ $N(\alpha(t_0))=\pm u$. Let $N(\alpha(t_0))=\delta u$

Since $\dot\alpha(t_0)\perp N_{(\alpha(t_0))}$ $\qquad \delta=\pm 1$.

$h''(t_0)=\ddot\alpha(t_0)\cdot u = \ddot\alpha\cdot\delta N = k\cdot\delta = k(\alpha(t_0))\cdot N(\alpha(t_0))\cdot u$.

(b) construct $\theta(t)$ as in Ex 11.15. By Ex 10.10 $\frac{d\theta}{dt}=k\circ\alpha$. Then rotation index is

$\frac{1}{2\pi}\int_{\dot\alpha}\eta = \frac{1}{2\pi}\left(\theta(t_0+\tau)-\theta(t_0)\right) = \frac{1}{2\pi}\int_{t_0}^{t_0+\tau}\frac{d\theta}{dt}dt = \int_{t_0}^{t_0+\tau}(k\circ\alpha)(t)dt$

Gauss map N of C is onto because: $\forall u\in S^1$, $h(t_1)=h(t_1+\tau)$ $\forall t_1, t_1+\tau\in\mathbb{R}$, τ is period

so there must be $t_0\in(t_1,t_1+\tau)$ s.t. $h'(t_0)=0$. So $N(\alpha(t_0))=\pm u$

Since $\alpha(t)$ is periodic. $h(t)$ must have both ~~maximum~~ minimum and ~~minimum~~ maximum say, t_0, t_0' resp.

$h'(t_0)=h'(t_0')=0$, $h''(t_0)\geq 0$ $h''(t_0')\leq 0$. But since $N=\pm u$, $N\cdot u\neq 0$. So $h''(t_0)\neq 0$

So $h''(t_0)>0$. Likewise, $h''(t_0')<0$. $h''(t_0)>0 \Rightarrow u\cdot N(\alpha(t_0))>0 \Rightarrow N(\alpha(t_0))=u$

$h''(t_0')<0 \Rightarrow u\cdot N(\alpha(t_0'))<0 \Rightarrow N(\alpha(t_0'))=-u$. So N is onto

(c) As $k>0$ $\int_{t_0}^{t}(k\circ\alpha)(t)dt$ monotonically increasing wrt t. Set $\theta(t)=\theta_0+\int_{t_0}^{t}\eta(\dot\alpha(\tau))d\tau$

then $\dot\alpha=(|\dot\alpha|\cos\theta(t), \sin\theta(t))$ As $N(c)=N(t_0)$, so $(\cos\theta(c),\sin\theta(c))=(\cos\theta_0,\sin\theta_0)$

So $\theta(c)=2n\pi+\theta_0$. But by Ex 10.10, $\int_{t_0}^{c}(k\circ\alpha)(t)dt=\int_{t_0}^{c}\frac{d\theta}{dt}dt=\theta(c)-\theta_0=2n\pi$

As $k>0$, $c>t_0$. so $n>0$. If $n=2$ then there is a $t_1\in(t_0,c)$ $\theta(t_1)=\theta_0+2\pi$

because $\theta(t)$ is continuous. But $(\cos\theta(t_1),\sin\theta(t_1))=(\cos\theta_0,\sin\theta_0)$, So $N(\alpha(t_1))=N(\alpha(t_0))$

But ~~by~~ that contradicts with the assumption that $N(t)\neq N(t_0)$ $\forall t\in(t_0,c)$

So $n=1$, i.e. $\int_{t_0}^{c}(k\circ\alpha)(t)dt=2\pi$.

By ~~definition~~ (b) $\int_{t_0}^{t_0+\tau}(k\circ\alpha)(t)dt$ = rotation index of $\alpha * 2\pi=\pm 2\pi$.

As $k>0$. It equals 2π, which in turn equals $\int_{t_0}^{c}(k\circ\alpha)(t)dt$.

As $k>0$ $c=t_0+\tau$. (a) has shown the Gaussian map in onto,

Now we've proven that $N(t)=N(t_0)$ iff $t=t_0+\tau\cdot n$. But τ is period of α.

So Gaussian map is injection, In sum, it is one-to-one.

11.20 (a) α_f is just one point a_0, so obviously $k(f)=0$ (construct $v=-a_0$, θ_v)

(b) $\alpha_f(t)=(a_n\cos nt, a_n\sin nt)$ Similar to Example 2 on Pg. 75, $k(f)=n$.

(c) Construct $\varphi: [0,2\pi]\times[0,1]\to\mathbb{R}^2-\{0\}$. $\varphi(t,u)=f(u(\cos t+i\sin t))$

↑ by $f(z)\neq 0$ $\forall z: |z|\leq 1$ $\qquad$ obviously continuous