$\varphi_0(t) = f(0)\neq 0$, So $k(\varphi_0(t)) = 0$.

$\varphi_1(t) = f(\cos t + i\sin t)$ $\quad k(f) = k(\varphi_1(t)) \overset{*}{=} k(\varphi_0(t)) = 0$ (*by Ex 11.17)

(d) Construct $\varphi(t,u) = \begin{cases} u^n f\left(\frac{1}{u}(\cos t + i\sin t)\right) & \text{if } u\neq 0 \\ a_n(\cos t + i\sin t)^n & \text{if } u = 0. \end{cases}$ on $[0,2\pi]\times[0,1]$

By def of φ and $f(z)\neq 0\ \forall z\in\mathbb{C}, |z|\geq 1$, we have $\varphi(t,u)\not\equiv 0$.

As $\lim_{u\to 0} u^n f\left(\frac{1}{u}(\cos t + i\sin t)\right) = \lim_{u\to 0} u^n \sum_{k=0}^{n} a_n(\cos kt + i\sin kt)\frac{1}{u^k} = a_n(\cos nt + i\sin nt)$
$= \varphi(t,0)$ So φ is continuous

$\varphi(t,0) = a_n(\cos nt + i\sin nt)$. By Example 2 on Pg 75. $k(\varphi(t,0)) = n$

By Ex 11.17. $k(f) = k(\varphi(t,1)) = k(\varphi(t,0)) = n$.

(e). Combining (c),(d), (c) says $k(f)=0$, (d) says $k(f) = n$. So ~~either~~ $n=0$

If no point of $\alpha(t)$ lies on positive x_1-axis, then choose $v=(1,0)$, by θ_v, we have $k(\alpha)=0$, correct.

11.21 Else Let $a < t_1 < t_2 < \cdots < t_m < b$ be the set of all $t\in(a,b)$ such that $\alpha(t)$ lies on the positive x_1-axis. we will discuss $t=a,b$ specially. ~~Note as $\alpha(a)=\alpha(b)$, even if $\alpha(a)$ is not on positive x_1-axis, we can still reparametrize $\alpha(t)$ into $\beta(t) = \alpha(t_1+t-a)$, then $\beta(a) = \alpha(t_1)$ which is on x_1-axis. If~~ Denote $t_0 = a$, $t_{m+1} = b$. For all $i = 1,2,\cdots,m$, if $\alpha(t_i)$ crosses positive x_1-axis upward, define $\delta_i = 1$. If crosses downward, define $\delta_i = -1$

If $\alpha(a)=\alpha(b)$ is on positive x_1-axis, define $\delta_0 = \delta_{m+1}$ likewise in $\{\pm 1\}$. ~~If $\alpha(a)$ is not on positive x_1-axis, define $\delta_0 = \delta_{m+1} = 0$~~.

$k(\alpha) = \frac{1}{2\pi}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\eta(\dot\alpha(t))dt = \frac{1}{2\pi}\sum_{i=0}^{m}\lim_{\varepsilon\to 0^+}\int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon}\eta(\dot\alpha(t))dt = \frac{1}{2\pi}\sum_{i=0}^{m}\lim_{\varepsilon\to 0^+}\int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon} d\theta_v(\alpha(t))dt$

where $v = (1,0)$ and θ_v is defined as in proof of Thm 3. We check two consecutive crossings of positive x_1-axis: $(i = 1,\cdots,m-1)$:

angle means $\lim_{\varepsilon\to 0^+}\left[\theta_v(\alpha(t_{i+1}-\varepsilon)) - \theta_v(\alpha(t_i+\varepsilon))\right]$.

i	$i+1$	δ_i	δ_{i+1}	angle	formula
↗	↗	1	+1	2π	
↗	↘	1	-1	0	$(\delta_i+\delta_{i+1})\cdot\frac{2\pi}{2}$
↘	↘	-1	-1	-2π	
↘	↗	-1	1	0	

So $k(\alpha) = \frac{1}{2}\sum_{i=1}^{m-1}(\delta_i + \delta_{i+1})$

if $\alpha(a)$ is not on pos x_1-axis

$+ \frac{1}{2\pi}\left(\theta_v(\alpha(b)) - 2\pi\cdot\frac{1-\delta_m}{2}\right)$

$+ \frac{1}{2\pi}\left(2\pi\frac{1+\delta_1}{2} - \theta_v(\alpha(b))\right)$

$= \sum_{i=1}^{m}\delta_i$

If $\alpha(a)$ is on pos x_1-axis, then $k(\alpha) = \frac{1}{2}\sum_{i=0}^{m}(\delta_i + \delta_{i+1}) = \sum_{i=0}^{m}\delta_i$ (as $\delta_{m+1} = \delta_0$)

So the conclusion is correct in both cases.

11.22 (a) let $\beta(t) = \alpha(t) - p$

$\eta(\dot\beta) = -\frac{\beta_2}{\beta_1^2+\beta_2^2}dx_1 + \frac{\beta_1}{\beta_1^2+\beta_2^2}dx_2 = -\frac{(\alpha_2(t)-b)\,dx_1}{(\alpha_1(t)-a)^2+(\alpha_2(t)-b)^2} + \frac{(\alpha_1(t)-a)\,dx_2}{(\alpha_1(t)-a)^2+(\alpha_2(t)-b)^2}$

So $k(\beta) = \int_\beta \eta(\beta) = \int_\alpha k_p(\alpha)$. We know $k(\beta) = 2k\pi$ $k\in\mathbb{Z}$, so $\frac{1}{2\pi}\int_\alpha k_p(\alpha)$ is integer

(b) Suppose p and q are joined by continuous $\beta: [c,d]\to\mathbb{R}^2 - \text{Image } \alpha$ s.t. $\beta(c) = p$. $\beta(d) = q$

Define $\varphi(t,u) = \alpha(t) - \beta(u)$ on $[a,b]\times[c,d]\to\mathbb{R}^2-\{0\}$ (Since $\beta\to\mathbb{R}^2 - \text{Image }\alpha$ so $\varphi\neq 0$)