Obviously, φ is continuous. $\varphi(t,c) = \alpha(t) - p$. $\varphi(t,d) = \alpha(t) - q$

$k(\varphi(t,c)) = k(\varphi(t,d)) \Rightarrow k_p(\alpha) = k_q(\alpha)$ (as $k_{\substack{p\\q}}(\alpha) = k(\alpha(t) - \substack{p\\q})$)

12.1 The matrix corresponding to L_p is $A = \|g\|^{-3}(g \cdot g^T - \|g\|^2 I)H$ ($g = \nabla f$, H is the Hessian)

~~$k(v)=$~~ $L_p(v) = -\|g\|^{-1} H \cdot V$, $k(v) = -\|g\|^{-1} V^T H V = \wp_p(v)$ for $V \in S_p$.

12.2 $\nabla f = (1,1,\dots,1)$, $V_i = \frac{1}{\sqrt{2}}(1,0\cdots0,-1,0\cdots0)$ where -1 is the i^{th} spot after 1, $i = 1 \cdots n$.

$\nabla f = \sqrt{n+1}$, $\frac{\partial^2 f}{\partial x_i \partial x_j} = 0$. So $k(v) = \wp_p(v) = 0$ $\forall V$ by Ex 12.1 So any ~~orth~~ $v \in S_p$ $\|v\| = 1$ is a principal curvature direction, with principal curvature 0.

$K(p) = 0$ $H(p) = 0$.

12.3 $\nabla f = (2x_1, \cdots, 2x_{n+1})$ $\|\nabla f(p)\| = 2r$ $H = 2 \cdot I$. $k(v) = \frac{-1}{r} v \cdot v$.

Any $v \in S_p$ $\|v\| = 1$ is a principal curvature direction, with principal curvature $\frac{-1}{r}$.

$k(p) = (-r)^{-n}$, $H(p) = \frac{-1}{r}$

12.4 $\nabla f = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}, \frac{2x_3}{c^2}\right)$ $H = \begin{pmatrix} 2a^{-2} & 0 & 0 \\ 0 & 2b^{-2} & 0 \\ 0 & 0 & 2c^{-2} \end{pmatrix}$ ~~$k(v)=$~~ $\|\nabla f(p)\| = \frac{2}{a}$ $\nabla f(p) = (\frac{2}{a}, 0, 0)$

$k(v) = -\frac{a}{2}\left(\frac{2}{a^2}v_1^2 + \frac{2}{b^2}v_2^2 + \frac{2}{c^2}v_3^2\right) = -a\left(\frac{v_1^2}{a^2} + \frac{v_2^2}{b^2} + \frac{v_3^2}{c^2}\right)$ $\forall v \in S_p$: $v = (0, v_2, v_3)$

$v_2^2 + v_3^2 = 1$, So $k(v)$ attains its extremum at $(0, \pm1, 0)$ and $(0,0,\pm1)$ i.e. principal curvature directions, corresponding to principal curvature $\frac{-a}{b^2}$ and $\frac{-a}{c^2}$ respectively.

$k(p) = a^2/b^2c^2$, $H(p) = \frac{-a}{2}\left(\frac{1}{b^2} + \frac{1}{c^2}\right)$

12.5 $\nabla f = \left(\frac{2x_1}{a^2}, \frac{2x_2}{b^2}, -\frac{2x_3}{c^2}\right)$ $H = 2\begin{pmatrix} a^{-2} & & \\ & b^{-2} & \\ & & -c^{-2} \end{pmatrix}$ $\|\nabla f(p)\| = \frac{2}{a}$ $\nabla f(p) = (\frac{2}{a}, 0, 0)$

$k(v) = -\frac{a}{2}\left(\frac{2}{a^2}v_1^2 + \frac{2}{b^2}v_2^2 - \frac{2}{c^2}v_3^2\right)$ $\forall v \in S_p$ $v = (0, v_2, v_3)$ $v_2^2 + v_3^2 = 1$ so

$k(v) = \frac{-a}{2}\left(\frac{2}{b^2}v_2^2 - \frac{2}{c^2}v_3^2\right) = -a\left(\frac{1}{b^2}(1 - v_3^2) - \frac{1}{c^2}v_3^2\right) = a\left[\left(\frac{1}{b^2} + \frac{1}{c^2}\right)v_3^2 - \frac{1}{b^2}\right]$ $v_3^2 \in [0,1]$

$k(v)$ attains max when $v_3^2 = 1$, max $= \frac{a}{c^2}$, attains min when $v_3^2 = 0$ min $= \frac{-a}{b^2}$

So principal curvature and principle curvature directions are: $((0,0,\pm1), \frac{a}{c^2})$, $((0,\pm1,0), \frac{-a}{b^2})$

$k(p) = -\frac{a^2}{b^2c^2}$, $H(p) = \frac{a}{2}\left(\frac{1}{c^2} - \frac{1}{b^2}\right)$

12.6 $\nabla f = \left(2x_1, 2x_2\left(1 - 2(x_2^2 + x_3^2)^{\frac{-1}{2}}\right), 2x_3\left(1 - 2(x_2^2 + x_3^2)^{\frac{-1}{2}}\right)\right)$

$$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 - 4x_3^2(x_2^2+x_3^2)^{-3/2} & 4x_2x_3(x_2^2+x_3^2)^{-3/2} \\ 0 & 4x_2x_3(x_2^2+x_3^2)^{-3/2} & 2 - 4x_2^2(x_2^2+x_3^2)^{-3/2} \end{pmatrix}$$

For (a) $\nabla f(p) = (0,2,0)$ $\|\nabla f\| = 2$, ($p = (0,3,0)$)

$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2/3 \end{pmatrix}$

(b) $p = (0,1,0)$, $\nabla f(p) = (0,-2,0)$ $\|\nabla f\| = 2$

$H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$