(a) $k(v) = -V_1^2 - V_2^2 - \frac{1}{3}V_3^2$, $V = (V_1, 0, V_3)$ $V_1^2 + V_3^2 = 1$, $k(v) = -1 + \frac{2}{3}V_3^2$

min $= -1$ when $V_3 = 0$. max $= -\frac{1}{3}$ when $V_3 = \pm 1$. So $((\pm 1, 0, 0), -1)$ and $((0,0,\pm 1), -\frac{1}{3})$

$H(p) = -\frac{2}{3}$, $K(p) = \frac{1}{3}$

(b) $k(V) = -V_1^2 - V_2^2 + V_3^2$ $V = (V_1, 0, V_3)$ $V_1^2 + V_3^2 = 1$ $k(V) = -1 + 2V_3^2$

min $= -1$ when $V_3 = 0$., max $= 1$ when $V_3 = \pm 1$

So $((\pm 1, 0, 0), -1)$, $((0,0,\pm 1), 1)$ $H(p) = 0$. $K(p) = -1$

12.7 If (λ_i, V_i) are eigenvalue vector of L_p for S. For $\forall v \in S_p$, $L_p(v) = -\nabla_v N = -(-\nabla_v(-N)) = -\tilde{L}_p(v)$ where $\tilde{L}_p$ stands for the Waingarten map for orientation $-N$. Thus Specifically $L_p(V_i) = \lambda_i V_i \iff \tilde{L}_p(V_i) = -\lambda_i V_i$. So L_p's eigenvalue λ_i corresponds to $\tilde{L}_p$'s eigenvalue $-\lambda_i$. So $\tilde{K} = \prod_{i=1}^{n}(-\lambda_i) = (-1)^n \prod_{i=1}^{n} \lambda_i = (-1)^n K$

12.8 As $n=2$, the Gaussian curvature is independent of orientation

Apply Thm 5. $Z = \frac{1}{2}\nabla f(p) = (p, x_1, x_2, -x_3)$ take $V_1 = (p, x_3, 0, x_1)$, $V_2 = (0, x_3, x_2)$

So $V_1 \perp Z$, $V_2 \perp Z$ $\det\begin{pmatrix}\nabla_{V_1} Z \\ \nabla_{V_2} Z \\ Z(p)\end{pmatrix} = \begin{vmatrix} x_3 & 0 & -x_1 \\ 0 & x_3 & -x_2 \\ x_1 & x_2 & -x_3 \end{vmatrix} = (x_1^2 + x_2^2 - x_3^2)x_3$

$\det\begin{pmatrix}V_1 \\ V_2 \\ Z(p)\end{pmatrix} = \begin{vmatrix} x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \\ x_1 & x_2 & -x_3 \end{vmatrix} = -x_3(x_1^2 + x_2^2 + x_3^2)$, $\|Z(p)\| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ $p + (t - t_0)\frac{p}{\|p\|}$

So $K(p) = x_3(x_1^2 + x_2^2 - x_3^2) / [(x_1^2 + x_2^2 + x_3^2) \cdot (-x_3)(x_1^2 + x_2^2 + x_3^2)] = 0$ $|\dot{\alpha}(t_0)| = 1$ $\ddot{\alpha} = 0$

This is because through each point p, there's a $\alpha(t)$ $(k(\dot{\alpha}(t_0)) = 0)$ which lie completely in S, so S doesn't force any acceleration. Besides, if S is oriented outward, then S always bends away from N, so $k(v) \le 0$. If oriented inward, then $k(v) \ge 0$. In whatever case, 0 is an extreme point of $k(v)$. So 0 is an eigenvalue of L_p. So $K(p) = 0$.

12.9 $Z = \frac{1}{2}\nabla f(p) = (p, x_1/a^2, x_2/b^2, -x_3/c^2)$ For $x_3 \ne 0$ we may take

$V_1 = (p, x_3/c^2, 0, x_1/a^2)$, $V_2 = (p, 0, x_3/c^2, x_2/b^2)$ $V_1, V_2 \perp Z$

$\det\begin{pmatrix}\nabla_{V_1} Z \\ \nabla_{V_2} Z \\ Z(p)\end{pmatrix} = \begin{vmatrix} x_3/a^2c^2 & 0 & -x_1/a^2c^2 \\ 0 & x_3/b^2c^2 & -x_2/b^2c^2 \\ x_1/a^2 & x_2/b^2 & -x_3/c^2 \end{vmatrix}$ $\det\begin{pmatrix}V_1 \\ V_2 \\ Z(p)\end{pmatrix} = \begin{vmatrix} x_3/c^2 & 0 & x_1/a^2 \\ 0 & x_3/c^2 & x_2/b^2 \\ x_1/a^2 & x_2/b^2 & -x_3/c^2 \end{vmatrix}$ $\|Z(p)\| = \left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^{1/2}$

$= \frac{x_3}{a^2b^2c^4}(x_1^2/a^2 + x_2^2/b^2 - x_3^2/c^2) = \frac{x_3}{a^2b^2c^4}$ $= \frac{-x_3}{c^2}\left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)$

$K(p) = -\left[a^2b^2c^2\left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^2\right]^{-1}$. negtitive At each point p, there are some directions bends towards N, some directions bending away from N. So the max of $k(v) > 0$. min $k(v) < 0$

As $K(p)$ = product of two extreme values, $K(p) < 0$

12.10. $Z = \frac{1}{2}\nabla f(p) = (p, \frac{2}{a^2}x_1, \frac{2}{b^2}x_2, -1)$, $V_1 = (p, 1, 0, \frac{2}{a^2}x_1)$ $V_2 = (p, 0, 1, \frac{2}{b^2}x_2)$