$$\det\begin{pmatrix}\nabla_{v_1}Z\\ \nabla_{v_2}Z\\ Z(p)\end{pmatrix} = \begin{vmatrix} 2/a^2 & 0 & 0\\ 0 & 2/b^2 & 0\\ 2x_1/a^2 & 2x_2/b^2 & -1\end{vmatrix} = -4/a^2b^2 \qquad \det\begin{pmatrix}V_1\\ V_2\\ Z(p)\end{pmatrix} = \begin{vmatrix} 1 & 0 & 2x_1/a^2\\ 0 & 1 & 2x_2/b^2\\ 2x_1/a^2 & 2x_2/b^2 & -1\end{vmatrix} = -1 - 4x_1^2/a^4 - 4x_2^2/b^4 \qquad \|Z(p)\|^2 = 1 + \frac{4x_1^2}{a^4} + \frac{4x_2^2}{b^4}$$

$\cancel{Z(p)}$ $K(p) = 4/a^2b^2\left(1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4}\right)^2$. $K(p) > 0$ As can be seen from the fact that S bends towards N at all points in all directions in S_p if S is inward oriented. If outward, then always bend away from N in all directions. So the product $K(p) > 0$.

12.11 $Z = \left(p, \frac{2x_1}{a^2}, \frac{-2x_2}{b^2}, -1\right)$, $V_1 = \left(p, 1, 0, \frac{2x_1}{a^2}\right)$, $V_2 = \left(p, 0, 1, \frac{-2x_2}{b^2}\right)$

$$\det\begin{pmatrix}\nabla_{v_1}Z\\ \nabla_{v_2}Z\\ Z(p)\end{pmatrix} = \begin{vmatrix} 2/a^2 & 0 & 0\\ 0 & -2/b^2 & 0\\ 2x_1/a^2 & -2x_2/b^2 & -1\end{vmatrix} = 4/a^2b^2, \quad \det\begin{pmatrix}V_1\\ V_2\\ Z(p)\end{pmatrix} = \begin{vmatrix} 1 & 0 & 2x_1/a^2\\ 0 & 1 & -2x_2/b^2\\ 2x_1/a^2 & -2x_2/b^2 & -1\end{vmatrix} = -1 - 4x_1^2/a^4 - 4x_2^2/b^4$$

$$\|Z(p)\|^2 = 1 + \frac{4x_1^2}{a^4} + \frac{4x_2^2}{b^4}$$

$K(p) = -4/a^2b^2\left(1+\frac{4x_1^2}{a^4}+\frac{4x_2^2}{b^4}\right)^2 < 0$ hard to plot and analyze its shape but look at the graph at ~~http://here~~ http://users.rsise.anu.edu.au/~xzhang/reading/ex1211.jpg

12.12 (a) Cylinder C: $g(x_1,x_2,x_3) = f(x_1,x_2)$, $Z = \nabla g(p) \underset{f(x,y)}{=} (f'_x, f'_y, 0)$,

$V_1 = (0,0,1)$. $V_2 = (f'_y, f'_x, 0)$ $\nabla_{v_1}Z = (0,0,0)$, So $\cancel{Z}$ $K(p) = 0$ by Thm 5.

(b) $g(x_1 \cdots x_{n+1}) = f(x_1 \cdots x_n)$ $\quad Z = \nabla g(p) = (f'_{x_1}, \cdots, f'_{x_n}, 0)$

$V_1 = (0, \cdots 0, 1)$ ~~V_2~~ and then decide $V_2 \cdots V_n$. $\nabla_{V_1}Z = (0, \cdots, 0)$ So $K(p) = 0$.

12.13 ~~$Z =$~~ $f = x_{n+1} - g(x_1 \cdots x_n)$ $\quad Z = \nabla f(p) = (-g'_1, \cdots, -g'_n, 1)$, (So $Z \cdot (0, 0 \cdots 0, 1) > 0$).

$V_1 = (1, 0, \cdots 0, g'_1), \cdots, V_n = (0, \cdots 0, 1, g'_n)$, $\nabla_{V_1}Z = (-g''_{11}, \cdots, -g''_{n1}, 0), \cdots, \nabla_{V_n}Z = (-g''_{n1}, \cdots, -g''_{nn}, 0)$

$$\det\begin{pmatrix}\nabla_{V_1}Z\\ \vdots \\ \nabla_{V_n}Z\\ Z(p)\end{pmatrix} = \begin{pmatrix} -g''_{11} & \cdots & -g''_{1n} & 0\\ \vdots & & \vdots & \vdots \\ -g''_{n1} & \cdots & -g''_{nn} & 0\\ -g'_1 & \cdots & -g'_n & 1\end{pmatrix} = (-1)^n \det\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right), \quad \det\begin{pmatrix}V_1\\ \vdots \\ V_n\\ Z(p)\end{pmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 & g'_1\\ & \ddots & & & \vdots \\ 0 & & \cdots & 1 & g'_n\\ -g'_1 & \cdots & & -g'_n & 1\end{vmatrix} = 1 + \sum_{i=1}^n \left(\frac{\partial g}{\partial x_i}\right)^2 = \|Z_p\|^2$$

↖ easy proof by induction

$$K(p) = (-1)^n \cdot (-1)^n \det\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right) \Big/ \left(1 + \sum_{i=1}^n \left(\frac{\partial g}{\partial x_i}\right)^2\right)^{\frac{n}{2}+1} = \det\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right) \Big/ \left(1 + \sum_{i=1}^n \left(\frac{\partial g}{\partial x_i}\right)^2\right)^{1+n/2}$$

12.14. ~~If $v \times w = 0$ then $\exists \lambda \in \mathbb{R}$ $v = \lambda w$, $L_p(v) \times L_p(w) = \lambda L_p(w) \times L_p(w) = 0 = K(p)\, v \times w$~~

~~Otherwise~~ Both $L_p(v) \times L_p(w)$ and $v \times w \in S_p^{\perp}$ (even if $v \times w = 0$ i.e. $v /\!/ w$). So to prove the result, one only needs to prove that $N(p) \cdot L_p(v) \times L_p(w) = N(p) \cdot v \times w$, where $N(p)$ is Gauss map. $\|N(p)\| = 1$

By Thm 5, $K(p) = \begin{vmatrix} -L_p(v)\\ -L_p(w)\\ N(p)\end{vmatrix} \Big/ \|N(p)\|^2 \begin{vmatrix} v\\ w\\ N(p)\end{vmatrix}$ So

$N(p) \cdot L_p(v) \times L_p(w) = \begin{vmatrix} L_p(v)\\ L_p(w)\\ N(p)\end{vmatrix} = K(p) \cdot \begin{vmatrix} v\\ w\\ N(p)\end{vmatrix} = K(p)$ ~~$v \times w$~~ $N(p) \cdot v \times w$.

12.15. By Thm 5. $K(p) = \begin{vmatrix} \nabla_v Z\\ \nabla_w Z\\ Z(p)\end{vmatrix} \Big/ \|Z(p)\|^2 \begin{vmatrix} v\\ w\\ Z(p)\end{vmatrix} = Z(p) \cdot \nabla_v Z \times \nabla_w Z \,/\, \|Z(p)\|^4$

as $\begin{vmatrix} v\\ w\\ Z(p)\end{vmatrix} = Z(p) \cdot v \times w = Z(p) \cdot Z(p) = \|Z(p)\|^2$

$\begin{vmatrix} \nabla_v Z\\ \nabla_w Z\\ Z(p)\end{vmatrix} = Z(p) \cdot v \times w$