12.16 By Thm 2, the eigenvectors of L comprises an orthonormal basis for S_p, let them be $(\alpha_1, \cdots \alpha_n)$, with corresponding eigenvalues $k_1(p) \cdots k_n(p)$. (Let $V = (V_1, \cdots V_n) = (\alpha_1 \cdots \alpha_n)T$ [A]. By Thm 3, ~~$k(V_i) = \sum_{j=1}^{n} k_j(p) t_i$~~ As $V_i = \sum_{j=1}^{n} \alpha_j t_{ji}$, so $k(V_i) = \sum_{j=1}^{n} k_j(p) t_{ji}^2$

So $\sum_{i=1}^{n} k(V_i) = \sum_{i,j=1}^{n} k_i(p) t_{ij}^2 = \sum_{i=1}^{n} k_i(p) \cdot \sum_{j=1}^{n} t_{ij}^2$, As Both V and A are orthonormal, $I = V^T V = T^T A^T A T = T^T T$, So T is also orthonormal. So $T T^T = I$ (I is identity)
So $\sum_{j=1}^{n} t_{ij}^2 = 1$ for all $i = 1 \cdots n$. So $\sum_{i=1}^{n} k(V_i) = \sum_{i=1}^{n} k_i(p)$, thus $H(p) = \frac{1}{n}\sum_{i=1}^{n} k_i(p) = \frac{1}{n}\sum_{i=1}^{n} k(V_i)$

12.17 (a) Obvious by Thm 3. Anyway $L(V(\theta)) = (\cos\theta) L(V_1) + (\sin\theta) L(V_2)$

$k(V(\theta)) = L(V(\theta)) \cdot V(\theta) = \cos^2\theta \cdot L(V_1) \cdot V_1 + \sin^2\theta \cdot L(V_2) V_2$
$+ \sin\theta\cos\theta\,(L(V_1) \cdot V_2 + L(V_2) \cdot V_1)$

$L(V_1) \cdot V_2 = k_1 \cdot V_1 \cdot V_2 = 0$. $L(V_2) \cdot V_1 = 0$.

So $k(V(\theta)) = \cos^2\theta \cdot k_1 + \sin^2\theta \cdot V_2$

(b) $H_p = \frac{1}{2}(k_1 + k_2)$, $\frac{1}{2\pi}\int_0^{2\pi} k(V(\theta))\,d\theta = \frac{1}{2\pi}\left[k_1 \int_0^{2\pi} \cos^2\theta\, d\theta + k_2 \int_0^{2\pi} \sin^2\theta\, d\theta\right] = \frac{1}{2}(k_1 + k_2) = H_p$.

12.18 $\operatorname{div} N = \operatorname{tr}(v \mapsto \nabla_v N) = \operatorname{tr}(-L_p)$ ~~$= -\operatorname{tr}(L_p)$~~

If $V_1 \cdots V_n$ are eigenvalues ($\lambda_1 \cdots \lambda_n$) of L_p. then $+V_1, \cdots +V_n$ are eigenvalues ($-\lambda_1 \cdots -\lambda_n$) of $-L_p$ because
$L_p(V_i) = \lambda_i V_i \iff -L_p(V_i) = -\lambda_i V_i$ So $\operatorname{tr}(-L_p) = \sum_{i=1}^{n}(-\lambda_i) = -\operatorname{tr}(L_p)$

So $H_p = \frac{1}{n}\operatorname{tr}(L_p) = \frac{-1}{n}\operatorname{div} N$

12.19 (a) $\tilde{S}$ is $g^{-1}(c)$. $\nabla g(p) = 0 \iff \frac{1}{a}\nabla f(p/a) = 0$

But S is n-surface, so $\nabla f(p/a) \neq 0$ for all p and thus $\nabla g(p) \neq 0$ $\forall p$, so $\tilde{S}$ is n-surface

$p \in S \iff f(p) = c \iff g(ap) = f(p) = c \iff ap \in \tilde{S}$

(b) $\forall N$ in the Gauss image of S, $\exists p$, s.t. $\nabla f / \|\nabla f\| \big|_p = N$. But $\nabla g(ap) = \frac{1}{a}\nabla f(p)$

$\nabla g / \|\nabla g\| \big|_{ap} = \frac{1}{a}\nabla f(p) / \|\frac{1}{a}\nabla f(p)\| = \nabla f / \|\nabla f\| = N$. So N is also in Gauss Image of $\tilde{S}$

$\forall N$ in Gauss image of $\tilde{S}$, $\exists q$ s.t. $\nabla g(q) / \|\nabla g(q)\| = N$. But $\nabla g(q) = \frac{1}{a}\nabla f(q/a)$

$\nabla f(q/a) / \|\nabla f(q/a)\| = a\nabla g(q) / \|a\nabla g(q)\| = \nabla g(q) / \|\nabla g(q)\| = N$.

So the spherical images of S and $\tilde{S}$ are the same

(c) $\forall v \in S_p$. $k(v) = -\nabla_v N \cdot v$, $\nabla_v N = (\nabla N_1(p) \cdot v, \cdots, \nabla N_{n+1}(p) \cdot v)^T \cdot v$

$\nabla N_i(p) = \left(\frac{\partial N_i}{\partial x_1}(p), \cdots, \frac{\partial N_i}{\partial x_{n+1}}(p)\right)$. As short hand, denote $\nabla f = (f'_1, \cdots, f'_{n+1})$

So $\frac{\partial N_i}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{f'_i}{\|\nabla f\|} = \frac{1}{\|\nabla f\|^2}\left(f''_{ij}\|\nabla f\| - f'_i \cdot \frac{1}{\|\nabla f\|}\sum_{k=1}^{n} f'_k f''_{kj}\right) = \frac{1}{\|\nabla f\|^3}\left(f''_{ij}\|\nabla f\|^2 - f'_i \sum_{k=1}^{n} f'_k f''_{kj}\right)\Big|_p$ (1)

$\forall v \in \tilde{S}_{ap}$, $\tilde{k}(v) = -\nabla_v \tilde{N} \cdot v$. Using similar notation

$\frac{\partial \tilde{N}_i}{\partial x_j} = \frac{1}{\|\nabla g\|^3}\left(g''_{ij}\|\nabla g\|^2 - g'_i \sum_{k=1}^{n} g'_k f''_{kj}\right)\Big|_p$ (2) But $g(p) = f(p/a)$.

So $\nabla g(p) = \frac{1}{a}\nabla f(p/a)$, i.e. $\nabla g(ap) = \frac{1}{a}\nabla f(p)$, $g''_{ij}(p) = \frac{1}{a^2} f''_{ij}(p/a)$, i.e. $g''_{ij}(ap) = \frac{1}{a^2} f''_{ij}(ap)$