by plugging into (11,12)

So $\frac{\partial \tilde{N}_i}{\partial x_j}\Big|_{ap} = \frac{1}{a}\frac{\partial N_i}{\partial x_j}\Big|_p$ So $\tilde{k}(v) = k(v)/a$, which is ~~also~~ true at all the (shared) stationary points. on $\|v\|=1$

But mean curvature H is the average of k at stationary points, thus $H(ap) = \frac{1}{a}H(p)$.

(d) K (Gauss-Kronecker curvature) is the product of $k(v)$ at stationary points

So $\tilde{K}(ap) = a^{-n}K(p)$

Remark Above argument based on stationary points is not strict enough, especially considering the multiplicity of L_p's eigenvalues. A better proof is: ~~$\forall v, w \in S_p$~~ As $\nabla g(ap) = \frac{1}{a}\nabla f(p)$

So $S_p = \tilde{S}_{ap}$. $\forall v, w \in S_p$ ($\tilde{S}_{ap}$) $L_p(v)\cdot w = \frac{1}{2}[k(v+w) - k(v) - k(w)]$

let $\tilde{L}_{ap}$ be Weingarten map on $\tilde{S}$ at ap $\rightarrow \tilde{L}_{ap}(v)\cdot w = \frac{1}{2}[\tilde{k}(v+w) - \tilde{k}(v) - \tilde{k}(w)] = \frac{1}{a}L_p(v)\cdot w$ as $\tilde{k}(\cdot) = k(\cdot)/a$

Since w is arbitrary in S_p, so $\tilde{L}_{ap}(v) = \frac{1}{a}L_p(v)$ $\forall v$. So each eigenvalue λ_i of L_p corresponds to the eigenvalue λ_i/a of $\tilde{L}_p$. As H and K are average / product of eigenvalues, we have

$H(ap) = \frac{1}{a}H(p)$ $\quad K(ap) = a^{-n}K(p)$

Proof: set $w = \tilde{L}_{ap}(v) - \frac{1}{a}L_p(v) \in S_p$

then one has $(\tilde{L}_{ap}(v) - \frac{1}{a}L_p(v))\cdot(\tilde{L}_{ap}(v) - \frac{1}{a}L_p(v)) = 0$. So $\tilde{L}_{ap}(v) = \frac{1}{a}L_p(v)$

13.1 If S is convex at p, then h_u ($u = N(p)$ Gauss map) attains local max/min at p. So $\mathcal{H}_p$ is semi-definite, so $\mathcal{L}_p = \pm\mathcal{H}_p$ is also semi-definite. But the Gauss Kronecker curvature, which is equal to the product of all eigenvalues of $\mathcal{L}_p$, is negative. As S_p is even dimensional, there must be one positive and one negative eigenvalue, which contradicts the fact that $\mathcal{L}_p$ is semi-definite. So S is not convex at p.

13.2 $\forall v, w \in S_p$ $\quad \nabla_v(\text{grad } h)\, w = \nabla_v(\nabla h - (\nabla h\cdot N)N)w = \nabla_v(\nabla h)w - (\nabla h\cdot N)(\nabla_v N\cdot w)$

$\nabla_w(\text{grad } h)\, v = \nabla_w(\nabla h - (\nabla h\cdot N)N)v = \nabla_w(\nabla h)v - (\nabla h\cdot N)(\nabla_w N\cdot v)$

We know L_p is self-adjoint, i.e., $\nabla_v N\cdot w = \nabla_w N\cdot v$. Besides,

$\nabla_v(\nabla h)w = v^T H w = w^T H v = \nabla_w(\nabla h)w$ $\quad$ So $\nabla_v(\text{grad } h)w = \nabla_w(\text{grad } h)v$, so self-adjoint

13.3. (a) $\Rightarrow$ If $\mathcal{Q}$ is pos Def, then $\forall$ eigenvector v (value λ), $\mathcal{Q}(v) = \lambda v$, $\mathcal{Q}(v)\cdot v = \lambda > 0$ as $\mathcal{Q}$ is Pos Def

$\Leftarrow$ We know that the eigenvectors $v_1 \cdots v_n$ make up an orthonormal basis on S_p. $\forall v \in S_p$.

Let $v = \sum_{i=1}^{n} a_i v_i$ $\quad \mathcal{Q}(v)\cdot v = \mathcal{Q}(\sum_{i=1}^{n} a_i v_i)\cdot\sum_{i=1}^{n} a_i v_i = \left(\sum_{i=1}^{n} a_i \mathcal{Q}(v_i)\right)\left(\sum_{i=1}^{n} a_i v_i\right) = \left(\sum_{i=1}^{n} a_i\lambda_i v_i\right)\left(\sum_{i=1}^{n} a_i v_i\right)$

$= \sum_{i=1}^{n} a_i^2\lambda_i \geq 0$ because $\lambda_i > 0$.

It is equal to 0 iff ~~$\lambda_i = 0$~~ $a_i = 0$, i.e. $v = 0$

(b) $\Leftarrow$ Since L is self-adjoint linear transformation, its associated matrix L is symmetric so it has two real valued eigenvalues λ_1, λ_2. $\det L > 0 \Rightarrow \lambda_1\lambda_2 > 0$ But if $\lambda_1 < 0, \lambda_2 < 0$, then L is negative definition, i.e. there can't be any v: $\mathcal{Q}(v) > 0$.

thus $\lambda_1 > 0, \lambda_2 > 0$.