$\Leftarrow$ by definition $Q(v) > 0$ for all $v \neq 0$. As Q is Pos Def. both eigenvalues are positive, thus $\det L = \lambda_1 \lambda_2 > 0$

(c) L is non-singular $\Leftrightarrow \det L = \prod_{i=1}^{n} \lambda_i \neq 0 \Leftrightarrow \lambda_i \neq 0$ (λ_i are eigenvalues)

Q is non-degenerate $\Leftrightarrow$

i.e. p is non-degenerate

$\mathcal{H}_p$ is non-degenerate $\Leftrightarrow L: v \mapsto \nabla_v(\operatorname{grad} h)$ is non-singular $\Leftrightarrow \lambda_i \neq 0 \Leftrightarrow \forall v \;\; \nabla_v(\operatorname{grad} h) \neq 0$

13.4 If h is height function or any function which has constant $\|\nabla h\|$, then

$(h \circ \beta)'(t) = \nabla h(\beta(t)) \cdot \dot\beta(t) \leq \|\nabla h(\beta(t))\| \cdot \|\dot\beta(t)\| = \|\nabla h(\alpha(t))\| \cdot \|\dot\alpha(t)\|$

$= \nabla h(\alpha(t)) \cdot \dot\alpha(t) = (h \circ \alpha)'(t)$.

So $h(\alpha(b)) = h(\alpha(a)) + \int_a^b (h\circ\alpha)'(t) \geq h(\beta(a)) + \int_a^b (h \circ \beta)'(t)\,dt = h(\beta(b))$

Equality holds iff $\nabla h(\beta(t)) = \lambda \dot\beta(t)$ $\lambda \geq 0$. But $\nabla h(\alpha(t)) = \nabla h(\beta(t))$.

So $\|\nabla h(\alpha(t))\| = \lambda \|\dot\beta(t)\| = \lambda \|\dot\alpha(t)\|$ But $\dot\alpha(t) = \nabla h(\alpha(t))$. So $\lambda = 1$

So $\nabla h(\beta(t)) = \dot\beta(t)$, i.e. β is also a gradient line passing thru $\alpha(a)$, but such a line is unique, so $\beta = \alpha$.

this $\tilde h$ is actually the h in the question

If $\|\nabla h\| = \text{const}$ is not guaranteed, WE FEEL that this proposition may not hold.

The following is a counter-example. let $\tilde h(x_1, x_2) = h(x_1)$ $f(x_1, x_2) = x_2$

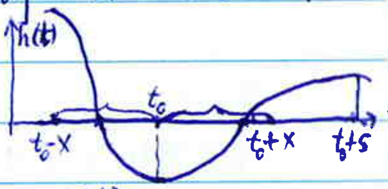
then $\nabla \tilde h = (h'(x_1), 0)$ $\nabla f = (0, 1)$ so $S = f^{-1}(0)$ is n-surface. $\nabla\tilde h \perp \nabla f \Rightarrow \operatorname{grad}\tilde h = \nabla \tilde h$

$\alpha(t), \beta(t) \in S$. So we can write in brief $\alpha(t) = (\alpha(t), 0)$. $\beta(t) = (\beta(t), 0)$

here $R \to R$ $\qquad R \to R$

So now $\dot\alpha(t) = h'(\alpha(t))$ $|\dot\beta(t)| = |\dot\alpha(t)|$

As $\beta(t)$ appears in the conclusion only inside $h(\beta(t))$, the only constraint on β is actually $\ell(\beta) = \int_a^b \|\dot\beta(t)\|dt = \int_a^b \|\dot\alpha(t)\|\,dt = \ell(\alpha)$. Now we check function $h(t) = \frac{1}{t}\sin\frac{1}{t}$ $(t > 0)$ so $\tilde h$ and f are defined on (R^+, R) which is open. let $\alpha(a) = t_0$, s.t. $\sin\frac{1}{t_0} = -1$



the first peak to the left of t_0 is $t_0 - X$, where $\frac{1}{t_0 - X} = \frac{1}{t_0} + \pi$, $X = \frac{\pi t_0^2}{1 + \pi t_0}$.

$h(t_0 - X) = \frac{1}{t_0} + \pi$, $h(t_0 + X) = \frac{1+\pi t_0}{t_0 + 2\pi t_0^2} \sin\frac{1+\pi t_0}{t_0 + 2\pi t_0^2} < \frac{1+\pi t_0}{t_0 + 2\pi t_0^2} < \frac{1+\pi t_0}{t_0} = h(t_0 - X)$.

Besides, the first peak to the right of t_0 is $t_0 + S$, s.t. $\frac{1}{t_0+S} = \frac{1}{t_0} - \pi$, $S = \frac{\pi t_0^2}{1 - \pi t_0} > X$

in (t_0, t_0+S) $\dot\alpha > 0$. Now suppose $\alpha(a) = t_0 + \varepsilon$ where $\varepsilon > 0$ is sufficiently small, . for $t > a$, $\alpha(t)$ monotonically increases. and $\beta(t)$ is forced to decrease monotonically. As ε can be arbitrarily small, by above discussion, $\beta(t)$ first reaches $t_0 - X$, while $\alpha(t)$ hasn't reached $t_0 + S$, i.e. $\dot\alpha = h \circ \alpha > h'(t_0 + \varepsilon)$ guarantees that α has enough impetus to go right and meanwhile, β goes left. β reaches $t_0 - X$ while α only reaches $t_0 + X + 2\varepsilon$.

Suppose b is chosen (stop) at such a moment, then we have $h(\alpha(b)) < h(\beta(b))$ which contradicts the exercise assertion.