13.5 $h(\beta(t))=c \Rightarrow \nabla h(\beta(t_1))\cdot\dot\beta(t_1)=0$, $\alpha(t_0)=\beta(t_1)$ $\Big\} \Rightarrow \Big\{$ $\nabla h(\alpha(t_0))\cdot\dot\beta(t_1)=0$, $\dot\alpha(t_0)=(\text{grad } h)(\alpha(t_0))$

~~$\Rightarrow (\text{grad } h)(\alpha(t_0))\cdot\beta(t_1)=0 \Rightarrow \nabla h$~~

$\Rightarrow \dot\alpha(t_0)\cdot\dot\beta(t_1) = (\text{grad } h)(\alpha(t_0))\cdot\dot\beta(t_1) = \big(\nabla h(\alpha(t_0)) - (\nabla h(\alpha(t_0))\cdot N(\alpha(t_0)))N(\alpha(t_0))\big)\cdot\dot\beta(t_1)=0$

$= \nabla h(\alpha(t_0))\cdot\dot\beta(t_1)=0$ As $N(\alpha(t_0))\cdot\dot\beta(t_1) = N(\beta(t_1))\cdot\dot\beta(t_1)=0$.

14.1 Let $S_1=f^{-1}(c)$. $S_2=g^{-1}(d)$, $\alpha(t): I\to S_1$, $\alpha(t_0)=p$. $\dot\alpha(t_0)=v$. As $\varphi(S_1)\subseteq S_2$, $g(\varphi(\alpha(t)))$ ~~$g(\alpha(t))$~~ $=d$

So $\nabla g(\varphi(\alpha(t)))\cdot(\varphi\circ\alpha)^{\cdot}(t)=0$. But $d\varphi(p,v)=(\varphi\circ\alpha)^{\cdot}(t_0)$, So $d\varphi(p,v)\perp\nabla g(\varphi(p))$, i.e.

$d\varphi(p,v)\in S_{2\,\varphi(p)}$ So $d\varphi: T(S_1)\to T(S_2)$

14.2 For $\forall p\in U_1, v\in\mathbb{R}^n$, $d(\psi\circ\varphi)(p,v)=(\psi(\varphi(p)), \nabla f_1(p)\cdot v,\dots,\nabla f_k(p)\cdot v)$, $f_i(p)=\psi_i(\varphi(p))$

$d\varphi(p,v)=(\varphi(p), \nabla\varphi_1(p)\cdot v, \dots \nabla\varphi_m(p)\cdot v)$. Let $u=(\nabla\varphi_1(p)\cdot v,\dots,\nabla\varphi_m(p)\cdot v)$

$d\psi\circ d\varphi(p,v)=(\psi(\varphi(p)), \nabla\psi_1(\varphi(p))\cdot u,\dots \nabla\psi_k(\varphi(p))\cdot u)$

But $\nabla\psi_i(\varphi(p))\cdot u = \sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))(\nabla\varphi_j(p)\cdot v) = \left(\sum_{j=1}^m \frac{\partial\psi_i}{\partial x_j}(\varphi(p))\cdot\nabla\varphi_j(p)\right)\cdot v$ and

$$\nabla f_i(p) = \left(\frac{\partial\psi_i}{\partial x_1}\cdots\frac{\partial\psi_i}{\partial x_m}\right)\begin{pmatrix}\frac{\partial\varphi_1}{\partial x_1}&\cdots&\frac{\partial\varphi_1}{\partial x_n}\\ \frac{\partial\varphi_m}{\partial x_1}&&\frac{\partial\varphi_m}{\partial x_n}\end{pmatrix} = \left(\sum_{j=1}^m\frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_1},\dots,\sum_{j=1}^m\frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_1}\right)=\sum_{j=1}^m\left(\frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_1},\dots,\frac{\partial\psi_i}{\partial x_j}\frac{\partial\varphi_j}{\partial x_n}\right)$$

$$=\sum_{j=1}^m\frac{\partial\psi_i}{\partial x_j}(\varphi(p))\cdot\left(\frac{\partial\varphi_j}{\partial x_1}(p),\dots,\frac{\partial\varphi_j}{\partial x_n}(p)\right)=\sum_{j=1}^m\frac{\partial\psi_i}{\partial x_j}(\varphi(p))\cdot\nabla\varphi_j(p)$$

So $d(\psi\circ\varphi)=d\psi\circ d\varphi$.

14.3. Example 9. $J^T=\begin{pmatrix}-\sin\theta & \cos\theta & \cancel{\sin\phi}\ 0 & \cancel{\cos\phi}\ 0\\ 0&0&-\sin\phi & \cos\phi\end{pmatrix}$ rank $J=2$.

Example 10. $J^T=\begin{pmatrix}\cos\frac\theta2\cos\theta & \cos\frac\theta2\sin\theta & \sin\frac\theta2\\ -\sin\theta-\frac t2\sin\frac\theta2\cos\theta - t\cos\frac\theta2\sin\theta, & \cos\theta-\frac t2\sin\frac\theta2\sin\theta+t\cos\frac\theta2\cos\theta, & \frac t2\cos\frac\theta2\end{pmatrix} \triangleq \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{pmatrix}$

$A\triangleq \begin{vmatrix}a_{11}&a_{13}\\a_{21}&a_{23}\end{vmatrix} = \frac t2(\cos\theta+\sin^2\theta)+\sin\frac\theta2\sin\theta$. $B\triangleq\begin{vmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{vmatrix} = \frac t2\sin\theta(1-\cos\theta)-\sin\frac\theta2\cos\theta$

If $A=B=0$ then $(\cos\theta+\sin^2\theta)\frac{-t}{2}=\sin\frac\theta2\sin\theta$ cross multiply $\times$ we have

$(\cos\theta-1)\sin\theta\cdot\frac{-t}{2}=\sin\frac\theta2\cos\theta$

$\frac t2\sin\frac\theta2\sin^2\theta(\cos\theta-1)=\frac t2\sin\frac\theta2\cos\theta(\cos\theta+\sin^2\theta)$ i.e. $\frac t2\sin\frac\theta2=0$. So $t=0$ or $\theta=2k\pi$ $k\in\mathbb{Z}$

If $t=0$ $J^T=\begin{pmatrix}\cos\frac\theta2\cos\theta & \cos\frac\theta2\sin\theta & \sin\frac\theta2\\ -\sin\theta&\cos\theta&0\end{pmatrix}$, $A^2+B^2+\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}^2=1$ So rank $J=2$.

If $\overset{t\neq0}{\theta}=2k\pi$, $J^T=\begin{pmatrix}\cos k\pi & 0 & 0\\ 0 & 1+t\cos k\pi & \frac t2\cos k\pi\end{pmatrix}$, $A^2+B^2+\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}^2=(1+t)^2+\frac14t^2>0$ So rank $J=2$.

In all, rank $J=2$ for all $t,\theta$.

14.4 Let $\alpha: I\to\mathbb{R}^2$ be a parametrized curve $\alpha(t)=\begin{vmatrix}\alpha_1(t)\\\alpha_3(t)\end{vmatrix}^{x_1}_{x_3}$, then the parametrized surface obtained by rotating about $x_3$-axis is $(\alpha_1(t)\cos\theta, \alpha_1(t)\sin\theta, \alpha_3(t))$. In Example 4 $\alpha(\phi)=\begin{pmatrix}r\sin\phi\\r\cos\phi\end{pmatrix}$.

Example 8 $\alpha(\phi)=\begin{pmatrix}a+b\cos\phi\\b\sin\phi\end{pmatrix}$