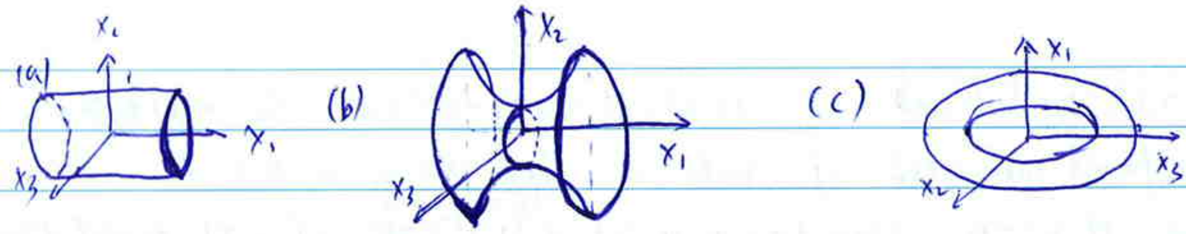
4.8 

ref Fig 14.5 Pg. 112 (change axis)

4.9 $f(x)=x_3^2+x_4^2-1$ $S=f^{-1}(0)$. $\nabla f=(0,0,2x_3,2x_4)$ If $\nabla f=0 \Rightarrow x_3=x_4=0 \Rightarrow$ not on S

4.10 $g(x_1,x_2,x_3,x_4)=f(x_1,x_2,(x_3^2+x_4^2)^{1/2})$

4.11 By Lagrange Thm, $\nabla g=\lambda\nabla f$. $\Rightarrow \begin{pmatrix}2x_1\\2x_2\end{pmatrix}=\lambda\begin{pmatrix}2ax_1+2bx_2\\2bx_1+2cx_2\end{pmatrix}\Rightarrow\lambda\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$
Since $ac-b^2>0\Rightarrow \det\begin{pmatrix}a&b\\b&c\end{pmatrix}\neq0 \Rightarrow\lambda\neq0\Rightarrow \begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\frac{1}{\lambda}\begin{pmatrix}x_1\\x_2\end{pmatrix}$
At that point $g=(x_1,x_2)\begin{pmatrix}x_1\\x_2\end{pmatrix}=\lambda(x_1,x_2)\begin{pmatrix}a&b\\b&c\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\lambda$. Note λ^{-1} is eigenvalue of $\begin{pmatrix}a&b\\b&c\end{pmatrix}$

4.12 $g=x^TAx$ $\nabla g=2Ax$ $f=\sum_{i=1}^{n}x_i^2$ $\nabla f=2x$ $\nabla g=\lambda\nabla f\Rightarrow Ax=\lambda x$
$g(x)=\lambda x^Tx=\lambda$ the eigenvalue of A

4.13 By Lagrange Thm, $\lambda\nabla f(P)=\nabla g(P)$. As $\nabla g(P)\neq0$ $\lambda\neq0$ $\forall v$: $v\cdot\nabla g(P)=0\Leftrightarrow v\cdot\nabla f(P)=0$
So tangent space of g through P is equal to tangent space of f through P

4.14 Let $g=\|x-P_0\|^2$. $S=f^{-1}(c)$. Since P is an extreme point of g on S
$\nabla g(P)=\lambda\nabla f(P)$ But $\nabla g(P)=2(P-P_0)$. So $(P,P-P_0)\perp S_P$.

4.15 $\nabla\det(X)=\frac{1}{\det(X)}(X^{-1})^T$ So $\nabla\det(X)=0$ is impossible.

4.16 (a) $\nabla\det(X)=\frac{1}{1}\begin{pmatrix}1&0\\0&1\end{pmatrix}^T=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. So $\left\langle\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\right\rangle_F=0\Rightarrow x_1+x_4=0$
(b) $\nabla\det(g)=\frac{1}{1}\begin{pmatrix}1&-1\\-1&2\end{pmatrix}$ So $SL(2)_g=\{(P,\begin{pmatrix}a&b\\c&d\end{pmatrix}):a-b-c+2d=0\}$.

4.17 (a) The proof in 4.15 is independent of dimension
(b) $\nabla\det(D)=I$. So $SL(3)_D=\{(P,M)\mid M\in R^{3\times3}, tr(M)=0\}$

5.1 ~~Only need to prove every point is connected to (0,0,1) Origin $\forall x$, define~~
$\forall x_1,x_2$, consider parametrized curve $\alpha(t)=x_1\cos t+u\sin t$ ~~where $u=\frac{e_1-x\cos\theta}{\sin\theta}$~~
~~then $\alpha(0)=x_1$~~, where $u=\frac{x_2-x_1\cos\theta}{\sin\theta}$, here $\theta=\cos^{-1}(x_1\cdot x_2)$ if $\sin\theta\neq0$ ~~here $\theta=\arccos(x$ (if $\sin\theta\neq0$)~~
then $\alpha(0)=x_1$ $\alpha(\theta)=x_1\cos\theta+\frac{x_2-x_1\cos\theta}{\sin\theta}\sin\theta=x_2$,
$\|u\|=\frac{1}{\sin^2\theta}(1+\cos^2\theta-2\cos\theta\langle x_1,x_2\rangle)=\frac{1}{\sin^2\theta}(1+\cos^2\theta-2\cos^2\theta)=1$, $\langle u,x_1\rangle=\frac{\langle x_1,x_2\rangle-\cos\theta}{\sin\theta}=\frac{\cos\theta-\cos\theta}{\sin\theta}=0$