14.5 (a) $J^T = \begin{pmatrix} \cos\phi\sin\theta\sin\psi & -\sin\phi\sin\theta\sin\psi & 0 & 0 \\ \sin\phi\cos\theta\sin\psi & \cos\phi\cos\theta\sin\psi & -\sin\theta\sin\psi & 0 \\ \sin\phi\sin\theta\cos\psi & \cos\phi\sin\theta\cos\psi & \cos\theta\cos\psi & -\sin\psi \end{pmatrix} \triangleq (A_1, A_2, A_3, A_4)$

$|A_1 A_2 A_3|^2 + |A_1 A_2 A_4|^2 + |A_1 A_3 A_4|^2 + |A_2 A_3 A_4|^2 = 1 + \sin^2\theta \sin^6\psi > 0$, So rank $J = 3$

(b) $(\sin\phi\sin\theta\sin\psi)^2 + (\cos\phi\sin\theta\sin\psi)^2 + (\cos\theta\sin\psi)^2 + \cos^2\psi = 1$

14.6 $J_\psi = \begin{pmatrix} t_{n+1} J_\phi & \begin{matrix} -a_1 \\ -a_{n+1} \end{matrix} \\ 0 & -a_{n+2} \end{pmatrix}$ $|J_\psi| = -a_{n+2} t_{n+1} |J_\phi| \neq 0$ (as $t \neq 0$, $a_{n+2} \neq 0$ $|J_\phi| \neq 0$ by assumption)

14.7 Let $d\varphi(v) = (\varphi(v), u) = (\varphi(v), \nabla\varphi_1 \cdot v, \cdots, \nabla\varphi_{n+k} \cdot v)$ Let $Y = X \circ \varphi$

$\nabla_v (X\circ\varphi) = (\nabla Y_1 \cdot v, \cdots \nabla Y_{n+k} \cdot v)$ $\nabla_{d\varphi(v)} X = (\nabla X_i \cdot u, \cdots, \nabla X_{n+k} \cdot u)$

But $\nabla X_i \cdot u = \sum_{j=1}^{n+k} \frac{\partial X_i}{\partial x_j} \nabla\varphi_j \cdot v = \left(\sum_{j=1}^{n+k} \frac{\partial X_i}{\partial x_j} \nabla\varphi_j \right) \cdot v$

$\nabla Y_i = \left(\frac{\partial X_i}{\partial x_1} \cdots \frac{\partial X_i}{\partial x_{n+k}} \right) \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \cdots & \frac{\partial\varphi_1}{\partial x_{n+k}} \\ \frac{\partial\varphi_{n+k}}{\partial x_1} & \cdots & \frac{\partial\varphi_{n+k}}{\partial x_{n+k}} \end{pmatrix} = \sum_{j=1}^{n+k} \frac{\partial X_i}{\partial x_j} \nabla\varphi_j =$ So $\nabla X_i \cdot u = \nabla Y_i \cdot v$

i.e. $\nabla_v (X\circ\varphi) = \nabla_{d\varphi(v)} X$

14.8 (a) $\|N\| = 1$. $\det \begin{pmatrix} E_1 \\ E_2 \\ N(p) \end{pmatrix} = \|E_1 \times E_2\| > 0$ as E_1, E_2 are linearly independent for parametrized 2-surface.

$N \perp E_1, N \perp E_2$, E_1, E_2 form a basis for $d\varphi_p$. So $N \perp$ Image $d\varphi_p$. So N is orientation vector field

As for uniqueness. $N \perp E_1, N \perp E_2 \Rightarrow N = \lambda \cdot E_1 \times E_2$, then $\|N\| = 1 \Rightarrow \lambda = \pm E_1 \times E_2$.

then $\det \begin{vmatrix} E_1 \\ E_2 \\ N \end{vmatrix} > 0 \Rightarrow N = E_1 \times E_2 / \|E_1 \times E_2\|$.

(b) E_1, E_2 are smooth wrt p as they are just the i-th column of ~~Jaco~~ Jacobian, So N is smooth

14.9 (a) Look at matrix $A = \begin{pmatrix} E_1 \\ \vdots \\ E_n \\ X \end{pmatrix} = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1n+1} \\ E_{n1} & E_{n2} & \cdots & E_{nn+1} \\ X_1 & X_2 & \cdots & X_{n+1} \end{pmatrix}$ Let $A_i = \begin{pmatrix} E_{11} & \cdots & E_{1,i+1} \cdots E_{1,n+1} \\ \vdots \\ E_{n1} & E_{n,i-1} & E_{n,i+1} \cdots E_{n,n+1} \end{pmatrix}$ $i = 1, 2, \cdots, n+1$ ~~So det A =~~

So $\det A = \sum_{i=1}^{n+1} (-1)^{n+1+i} \det A_i X_i = \sum_{i=1}^{n} (\det A_i)^2$ as $X_i = (-1)^{n+1+i} \det A_i$.

So $\det A \geq 0$ and $\det A = 0$ iff $\det A_i = 0$ for all $i = 1 \cdots n+1$. But that contradicts the fact that φ is a parametrized n-surface, i.e. Jacobian is non-singular. So $\det A > 0$

If $X(p) = 0$, then $|A| = 0$ which is impossible. Hence $X(p) \neq 0$ for all $p \in U$.

(b) For $i = 1 \cdots n$, $E_i \cdot X = \det \begin{pmatrix} E_1 \\ \vdots \\ E_i \end{pmatrix} = 0$ So $X \perp E_i$ So $X \perp$ Image $d\varphi_p$. So X is normal vector field along φ

(c) Combining (b), $\det A > 0$, and $\|N\| = 1$. We have N is orientation vector field along φ.

(d) X_i is ~~continuous~~ smooth and $X(p) \neq 0$. So N is smooth.

14.10 $E_i(p) = (\varphi(p), 0, \cdots, 1, \cdots 0, \frac{\partial g}{\partial u_i}(p))$. So $E_i(p) \cdot N(p) = 0$ Let $A = \begin{pmatrix} E_1 \\ E_n \\ N \end{pmatrix}$ ~~then~~ let $a_i = \frac{\partial g}{\partial u_i}(p)$ before normalization, then

$\det A = \begin{bmatrix} 1 & & a_1 \\ & \ddots & a_n \\ -a_1 \cdots & -a_n & 1 \end{bmatrix} = 1 + \sum_{i=1}^{n} a_i^2 > 0$. $\|N\| = 1$ So N is orientation vector field along φ