14.11 Proof is essentially similar to proving Thm 2 in Chapter 9. Let $v, w \in R^n_p$ and orientation N

We need to prove $L_p(d\varphi_p(v))\cdot d\varphi_p(w) = L_p(d\varphi_p(w))\cdot d\varphi_p(v)$ i.e. $+\nabla_v N\cdot d\varphi_p(w) = \nabla_w N\, d\varphi_p(v)$

Let X be the one defined in Ex 14.9. then

$$\nabla_v N\cdot d\varphi_p(w) = \nabla_v\left(\frac{X}{\|X\|}\right)\cdot d\varphi_p(w) = \left[(\nabla_v X)\frac{1}{\|X\|} + \left(\nabla_v \frac{1}{\|X\|}\right)X\right] d\varphi_p(w) = \frac{1}{\|X\|}\nabla_v X\cdot d\varphi_p(w) = v^T J_X^T(p)\, J_{\varphi}^{(p)} w/\|X(p)\|$$

Similarly $\nabla_w N\cdot d\varphi(v) = w^T J_N^T(p) J_\varphi(p)\, v/\|X(p)\|$ So we only need to prove that $J = J_X^T J_\varphi$ is symmetric

But $J_{ij} = \sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$ $\quad J_{ji} = \sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_j}\frac{\partial \varphi_k}{\partial x_i}$. Let $J_\varphi = (J_{\varphi 1}, J_{\varphi 2},\dots,J_{\varphi n})$. by def Ex 14.9.

$X\perp J_{\varphi i}$ $(i=1\cdots n)$ So $X\cdot J_{\varphi i}=0$ Taking derivative $J_X^T J_{\varphi i} + H_{\varphi i}^T X = 0$ where

$$H_{\varphi i} = \begin{pmatrix} \frac{\partial^2\varphi_1}{\partial x_i\partial x_1} & \cdots & \frac{\partial^2\varphi_1}{\partial x_i \partial x_n}\\ \vdots & & \\ \frac{\partial^2\varphi_{n+1}}{\partial x_i\partial x_1}&\cdots&\frac{\partial^2\varphi_{n+1}}{\partial x_i\partial x_n}\end{pmatrix}$$

So we have for $j=1\cdots n$. $\sum_{k=1}^{n+1} \frac{\partial X_k}{\partial x_j}\frac{\partial\varphi_k}{\partial x_i} + \sum_{k=1}^{n+1} \frac{\partial^2\varphi_k}{\partial x_i\partial x_j}X_k = 0$

Similarly: $\sum_{k=1}^{n+1} \frac{\partial X_k}{\partial x_i}\frac{\partial \varphi_k}{\partial x_j} + \sum_{k=1}^{n+1} \frac{\partial^2\varphi_k}{\partial x_j \partial x_i} X_k = 0$.

As $\frac{\partial^2\varphi_k}{\partial x_i\partial x_j} = \frac{\partial^2\varphi_k}{\partial x_i\partial x_j}$ So $\sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_j}\frac{\partial\varphi_k}{\partial x_i} = \sum_{k=1}^{n+1}\frac{\partial X_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$, i.e. $J_{ij} = J_{ji}$.

14.12 $d\varphi(p,v) = (\varphi(p), \nabla_v\varphi_1, \cdots \nabla_v\varphi_{n+k})$, $J_{d\varphi} = \begin{pmatrix} J_\varphi(p) & 0\\ sth. & \begin{matrix}\nabla\varphi_1\\ \vdots \\ \nabla\varphi_{n+k}\end{matrix}\end{pmatrix} = \begin{pmatrix} J_\varphi(p) & 0 \\ sth & J_\varphi(p)\end{pmatrix}$ As $J_\varphi(p)$ is full ranked, $J_{d\varphi(p,v)}$ must be full ranked as well

14.13 $\nabla_{e_i}E_j = \nabla_{e_i}\left(\varphi(p), \frac{\partial\varphi_1}{\partial x_j},\cdots,\frac{\partial\varphi_{n+k}}{\partial x_j}\right) = \left(\frac{\partial^2\varphi_1}{\partial x_i\partial x_j}, \frac{\partial^2\varphi_2}{\partial x_i\partial x_j},\dots,\frac{\partial^2\varphi_{n+k}}{\partial x_i \partial x_j}\right) = \nabla_{e_j}E_i$

14.14 (a) Let $\begin{pmatrix} L_p(E_1(p))\\ \vdots \\ L_p(E_n(p))\\ N(p)\end{pmatrix} = A^T \cancel{\begin{pmatrix}E_1(p)\\ \vdots\\ E_n(p)\end{pmatrix}}, \begin{pmatrix}A^T & \\ & 1\end{pmatrix}\begin{pmatrix}E_1(p)\\ \vdots\\ E_n(p)\\N(p)\end{pmatrix}$ By definition of $N(p)$, $\begin{pmatrix}E_1(p)\\E_n(p)\\N(p)\end{pmatrix}\neq 0$, so

$\det A = \det\begin{pmatrix}L_p(E_1(p))\\ L_p(E_n(p))\\ N(p)\end{pmatrix} \Big/ \det\begin{pmatrix}E_1(p)\\E_n(p)\\N(p)\end{pmatrix}$, as $L_p(E_i(p)) = -\nabla_{e_i}N = -\left(\varphi(p), \frac{\partial N_1}{\partial x_i}(p),\dots,\frac{\partial N_{n+1}}{\partial x_i}(p)\right)$

$= (-1)^n \det\begin{pmatrix}J_N^T\\N\end{pmatrix}\Big/\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix}$ (1)

On the other hand $L_p[E_i(p)]\cdot E_j(p) = -\nabla_{e_i}N\cdot E_j(p) = -\left(\frac{\partial N_1}{\partial x_i}\cdots\frac{\partial N_{n+1}}{\partial x_i}\right)\cdot E_j(p) = -\sum_{k=1}^{n+1} \frac{\partial N_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$ (*)

$E_i(p)\cdot E_j(p) = \sum_{k=1}^{n+1} \frac{\partial\varphi_k}{\partial x_i}\frac{\partial\varphi_k}{\partial x_j}$

So $\det(L_p[E_i(p)]\cdot E_j(p)) / \det(E_i(p)\cdot E_j(p)) = (-1)^n \det(J_N^TJ_\varphi)/\det(J_\varphi^TJ_\varphi)$ (2)

Notice $\begin{pmatrix}J_N^T\\N\end{pmatrix}(J_\varphi\ N^T) = \begin{pmatrix}J_N^TJ_\varphi & J_N^TN^T\\ NJ_\varphi & NN^T\end{pmatrix}$ By definition of N, $NN^T=1$. $NJ_\varphi=0$ we have

$\det\begin{pmatrix}J_N^T\\N\end{pmatrix}\cdot\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix} = \det(J_N^TJ_\varphi)$ (3)

Likewise $\begin{pmatrix}J_\varphi^T\\N\end{pmatrix}(J_\varphi\ N^T) = \begin{pmatrix}J_\varphi^TJ_\varphi & J_\varphi^TN^T\\NJ_\varphi & NN^T\end{pmatrix}$ hence $\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix}\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix} = \det(J_\varphi^TJ_\varphi)$ (4)

$\frac{(3)}{(4)}$ we have $\det\begin{pmatrix}J_N^T\\N\end{pmatrix}\Big/\det\begin{pmatrix}J_\varphi^T\\N\end{pmatrix} = \det(J_N^TJ_\varphi)/\det(J_\varphi^TJ_\varphi)$ then by (1), (2) we prove

$K(p) = \det A = \det[L_p[E_i(p)]\cdot E_j(p)] / \det(E_i(p)\cdot E_j(p))$

(b) $\cancel{\nabla_{e_i}E_j = \nabla_{e_i}\left(\frac{\partial\varphi_1}{\partial x_j} \cdots \frac{\partial\varphi_{n+1}}{\partial x_j}\right) = \left(\frac{\partial^2\varphi_1}{\partial x_j\partial x_1}\cdots\frac{\partial^2\varphi_{n+1}}{\partial x_j\partial x_i}\right)}$ As $E_j\cdot N=0$ we have

$0 = \nabla_{e_i}(E_j N) = \nabla_{e_i}E_j\, N + E_j(p)\nabla_{e_i}N$, So $\nabla_{e_i}E_j\cdot N(p) = (-\nabla_{e_i}N)\cdot E_j(p) = L_p(E_i(p))\cdot E_j(p)$

So $\det[L_p[E_i(p)]\cdot E_j(p)] / \det(E_i(p)\cdot E_j(p)) = \det[\nabla_{e_i}E_j\, N(p)] / \det(E_i(p)\cdot E_j(p))$.

So if n = even number, then whether using N or $-N$ doesn't matter