15.7 (a) To be in $O(2)$, the matrix must satisfy: $\begin{array}{l} x_1^2+x_2^2-1=0 \\ x_1x_3+x_2x_4=0 \\ x_3^2+x_4^2-1=0 \end{array}$ $J=\begin{pmatrix} 2x_1 & 2x_2 & 0 & 0 \\ x_3 & x_4 & x_1 & x_2 \\ 0 & 0 & 2x_3 & 2x_4 \end{pmatrix} = (J_1\ J_2\ J_3\ J_4)$ ~~if $(\alpha_1, \alpha_2\ \alpha_3) J = 0$~~

$Q = (\det(J_1 J_2 J_3))^2 + (\det(J_1 J_3 J_4))^2 + (\det(J_2 J_3 J_4))^2 + (\det(J_1 J_2 J_4))^2$

$= 16(x_1x_4 - x_2x_3)^2 \sum_{i=1}^{4} x_i^2$ if $=0$ So ① $\cancel{x}\ 0$, $\sum x_i^2 = 0 \Rightarrow x_i = 0 \Rightarrow x_1^2+x_2^2-1 \neq 0$ contradiction!

② $x_1x_4 = x_2x_3$, So $x_1x_4x_3 = x_2x_3^2$, i.e. $-x_2^2x_4^2 = x_2x_3^2$ So $x_2 = 0 \Rightarrow x_1 = \pm 1 \Rightarrow x_4 = 0$

$\Rightarrow x_3 = \pm 1 \Rightarrow x_1x_3 = \pm 1$ but $x_2x_4 = 0 \Rightarrow x_1x_3 + x_2x_4 \neq 0$ contradiction

So $Q \neq 0$, ~~is~~ $\text{rank}(J) = 3$, $O(2)$ is 1-surface in R^4.

(b) Now $J = \begin{pmatrix} 2&0&0&0 \\ 0&1&1&0 \\ 0&0&0&2 \end{pmatrix}$ $O(2)_p = \left\{(a,b,c,d) \,\middle|\, J\cdot\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix} 2&0&0&0 \\ 0&1&1&0 \\ 0&0&0&2 \end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = 0\right\} = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \,\middle|\, \begin{array}{l} a=d=0 \\ b+c=0\end{array}\right\}$

Solution 2 Let $\alpha(t) = \begin{pmatrix}\alpha_1(t)\\ \alpha_2(t)\end{pmatrix}$, then $\alpha(t) \in O(2) \Leftrightarrow \|\alpha_i(t)\| = 1$, $\alpha_i(t)\cdot\alpha_j(t) = 0$.

So $\alpha_i'\cdot\alpha_i = 0$ So $(a\ b)\begin{pmatrix}1\\0\end{pmatrix} = 0 \Leftrightarrow a = 0$. $(c\ d)\begin{pmatrix}0\\1\end{pmatrix} = 0 \Leftrightarrow d = 0$.

$\alpha_i'\cdot\alpha_j + \alpha_i\cdot\alpha_j' = 0 \Leftrightarrow (a\ b)\begin{pmatrix}0\\1\end{pmatrix} + (c\ d)\begin{pmatrix}1\\0\end{pmatrix} = 0 \Leftrightarrow b+c=0$

$\alpha(t_0) = \begin{pmatrix}\alpha_1(t_0)\\ \alpha_2(t_0)\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ Let $\alpha'(t_0) = \begin{pmatrix}a&b\\c&d\end{pmatrix}$

So $O(2)_p = \left\{\begin{pmatrix}a&b\\c&d\end{pmatrix} \,\middle|\, \begin{array}{l} a=d=0 \\ b+c=0\end{array}\right\}$.

15.8 (a) Prove that J has rank $\frac{1}{2}n(n+1)$ by induction on n. For $n=2$ 15.7 has proven it.

Let the $n\times n$ matrix be written as $\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}$, the constraints are $\|\alpha_i\|^2 = 1$, $\alpha_i\alpha_j = 0$ $(i\neq j)$. So Jacobian is

$\text{rank}\, J_n = \text{rank}\begin{pmatrix} 2\alpha_1 & & & \\ & 2\alpha_2 & & \\ & & \ddots & 2\alpha_n \\ \alpha_2 & \alpha_1 & & \\ \alpha_n & \cdots & & \alpha_1 \\ & & \alpha_n & \alpha_{n-1} \end{pmatrix} = \text{rank}\begin{pmatrix} J_{n-1} & 0 \\ 0 & 2\alpha_n \\ \alpha_n & \alpha_1 \\ \ddots & \vdots \\ \alpha_n & \alpha_{n-1} \end{pmatrix}$ ~~$= \text{rank}\begin{pmatrix} J_{n-1} & 0 \\ 0 & \alpha_n \\ \alpha_1 & 0 \\ \vdots & \vdots \\ \alpha_{n-1} & 0 \end{pmatrix}$~~ $\frac{n(n+1)}{2}$ rows, n^2 column

Note the lowest n rows are linearly independent. As If $\exists\, \beta_1 \cdots \beta_n \in R$ s.t.

$\beta_n\begin{pmatrix}0\\ \vdots\\ 0\\ \alpha_n^T\end{pmatrix} + \beta_1\begin{pmatrix}\alpha_n^T\\ 0\\ \vdots \\ 0\\ \alpha_1^T\end{pmatrix} + \beta_2\begin{pmatrix}0\\ \alpha_n^T\\ \vdots\\ 0\\ \alpha_2^T\end{pmatrix} + \cdots + \beta_{n-1}\begin{pmatrix}0\\ \vdots\\ 0\\ \alpha_n^T\\ \alpha_{n-1}^T\end{pmatrix} = 0$ So $\beta_i\alpha_i^T = 0$ $i = 1..n-1$ ①

$\sum_{i=1}^{n}\beta_i\alpha_i^T = 0$ ②

As none of the α_i is straight 0, $\beta_i = 0$ for $i = 1..n-1$ by ①. Then by ② $\beta_n\alpha_n^T = 0$ So $\beta_n = 0$.

Finally the rows in $(J_{n-1}\ 0)$ (the first $\frac{n(n-1)}{2}$ rows) are independent of the last n rows, because these $\frac{n(n-1)}{2}$ rows all have last n elements straight 0 and none of α_i is straight 0. ① $(\alpha_1, \cdots, \alpha_n)$ are linearly independent

So $\text{rank}\, J_n = n + \text{rank}\, J_{n-1} = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$. So $O(n)$ is $\frac{n(n-1)}{2}$ surface in R^{n^2}

(b) Let $\beta = \begin{pmatrix}\beta_1\\ \vdots\\ \beta_n\end{pmatrix} \in O(n)_p$ then $J\beta = 0$, i.e. $\left.\begin{array}{ll}\alpha_i\cdot\beta_i = 0 & \Rightarrow \beta_{ii} = 0 \\ \alpha_i\beta_j + \alpha_j\beta_i = 0 & \Rightarrow \beta_{ji} + \beta_{ij} = 0\end{array}\right\} \Rightarrow O(n)_p = \{P \in R^{n\times n} \mid \forall i, j.\ P_{ij} + P_{ji} = 0\}$

If we use the hint in Ex 15.7(b). $\alpha(t) \in O(n)$, $\|\alpha_i(t)\| = 1$, $\alpha_i(t)\cdot\alpha_j(t) = 0$

So $\alpha_i(t)\cdot\alpha_i'(t) = 0$, $\alpha_i(t)\alpha_j'(t) + \alpha_i'(t)\alpha_j(t) = 0$

i^{th} element of $\alpha_i'(t) = 0$, i^{th} element of $\alpha_j'(t)$ + j^{th} element of $\alpha_i'(t) = 0$

which yields the same result/conclusion.

This is because if $\sum_{j=1}^{n}\beta_j\alpha_j = 0$, $\beta_i \in R$, then $0 = \alpha_i\cdot\sum\beta_j\alpha_j = \beta_i\alpha_i\alpha_i = \beta_i$

15.9 $v \in S_p = \{v \in R_p^{n+1} \mid \nabla f_i(p)\cdot v = 0\} \Leftrightarrow \nabla f_i(p)\cdot v = 0\ \forall i \Leftrightarrow v \in \text{Ker}\, df_p$.