15.10 (brief proof). Since $J = \begin{pmatrix}\nabla f_1\\ \vdots \\ \nabla f_k\end{pmatrix}$ is fully ranked, so there are k independent columns indexed by $i_1 \cdots i_k$, which form matrix P. So $\det P \neq 0$. Define $\Psi(x_1 \cdots x_n)$ as $\Psi(x_1 \cdots x_{i_1-1}, f_1(x_1 \cdots x_{n+1}), x_{i_1+1}, \cdots, x_{i_k-1}, f_k(x_1 \cdots x_n), x_{i_k+1}, \cdots x_{n+1})$, whose Jacobian J satisfies $\det(J) = \det(P) \neq 0$. Then go on as in proof of Thm 1 by applying inverse function theorem. Finally, $U = \{(u_1 \cdots u_n) \in R^n \mid a_j < u_j < b_j$ for $j < i_1$, $a_{j+1} < u_j < b_{j+1}$ for $j \in [i_1, i_2)$, $\cdots$, $a_{j+k-1} < u_j < b_{j+k-1}$ for $j \in [i_{k-1}, i_k)$, $a_{j+k} < u_j < b_{j+k}$ for $j \geq i_k\}$ and define $\varphi: U \to R^{n+k}$ by
$\varphi(u_1 \cdots u_n) = (\Psi|_V)^{-1}(u_1 \cdots u_{i_1-1}, c_1, u_{i_1+1}, \cdots, u_{i_k-1}, c_k, u_{i_k+1}, \cdots, u_n)$. (elsewhere, just change $n+1$ to $n+k$ in proof of Thm 1)

15.11 (brief proof). Define $U \times R^k \to R^{n+k}$ by $\psi(q, t_1 \cdots t_k) = \varphi(q) + \sum_{i=1}^{k} t_i N_i(q)$, where N_i are the vector fields along φ which span the normal space $(\text{Image } d\varphi_q)^\perp$ for each $q \in U$. Then Jacobian $J\psi(p, 0 \cdots 0) = (J\varphi(p), N_1(p) \cdots N_k(p))$ whose determinant $\neq 0$. By the inverse func Thm, there is an open set $V \subset U \times R^k$ about $(p, \overbrace{0 \cdots 0}^{k \text{ 0's}})$ such that the restriction $\psi|_V$ of ψ to V maps V one to one onto the open set $\psi(V)$, and $(\psi|_V)^{-1}$ is smooth. By shrinking V if necessary, we may assume $V = U_1 \times I^k$ for some open set $U_1 \subseteq U$ containing p and some interval $I \subseteq R$ containing 0. Now define f: Image $\psi|_V \to R^k$ by $f(\psi(q, t_1 \cdots t_k)) = (t_1, \cdots, t_k)$. f is well defined and is smooth because f is the composition of the smooth map $(\psi|_V)^{-1}$ and projection map $U_1 \times I^k \to I^k$. The level set $f^{-1}(0 \cdots 0)$ is just $\varphi(U_1)$ because $f^{-1}(0) = \{\psi(q, t_1 \cdots t_k) \mid q \in U_1, t_i = 0\} = \{\varphi(q) \mid q \in U_1\}$. Finally we prove that $Jf(\delta)$ is fully ranked for $\delta = \psi(q, 0 \cdots 0) \in f^{-1}(0 \cdots 0)$. Let $\alpha_i(s) = \varphi(q) + s \cdot N_i(q)$ then $\nabla f_j(\delta) \cdot N_i(q)$
$= \nabla f_j(\alpha_i(0)) \cdot \dot{\alpha}_i(0) = \cancel{f_j}\, (f_j \circ \alpha_i)^{\cdot}(0) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. So

$\underbrace{\begin{pmatrix}\nabla f_1(\delta) \\ \vdots \\ \nabla f_k(\delta)\end{pmatrix}}_{\triangleq A} \cdot \underbrace{(N_1(q) \cdots N_k(q))}_{\triangleq B} = I_k$. ~~By definition rank $(N_1(q) \cdots N_k(q)) = k$~~ To be fast, let's quote a matrix result: $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

Now $k \leq \min(\text{rank}(A), \text{rank}(B))$. But both $\text{rank}(A)$ and $\text{rank}(B) \leq k$ thus $\text{rank}(A) = \text{rank}(B) = k$, i.e., A (Jacobian) is fully ranked.

To prove rank $Jf(\delta) = k$, another way is: assume $\sum_{i=1}^{k} \beta_i \nabla f_i(\delta) = 0$, then $\sum_{i=1}^{k} \beta_i (\nabla f_i(\delta) B) = 0$. But $\nabla f_i(\delta) \cdot B$ is just the i^{th} row of I_k. So $\beta_i = 0$ for all $i = 1 \cdots k$, i.e. $\{\nabla f_1(\delta), \cdots, \nabla f_k(\delta)\}$ are independent. Thus $\varphi(U_1) = f^{-1}(0 \cdots 0)$ is an n-surface in R^{n+k}

~~15.12~~

15.12 (a) $\varphi(p + tv) = \left(2(x_1 + tv_1), \cdots, 2(x_n + tv_n), \sum_{i=1}^{n}(x_i + tv_i)^2 - 1\right) / \left(1 + \sum_{i=1}^{n}(x_i + tv_i)^2\right)$

$\frac{d}{dt}\big|_0 \varphi(p + tv) = \left(2v_i\left(\sum_{j=1}^{n} x_j^2 + 1\right) - 4x_i \sum_{j=1}^{n} x_j v_j \text{ for } i = 1 \cdots n,\ -4\sum_{i=1}^{n} x_i v_i\right) / \left(\sum_{j=1}^{n} x_j^2 + 1\right)^2$

So $\|d\varphi(v)\|^2 = \|\frac{d}{dt}\big|_0 \varphi(p+tv)\|^2 = 4\left\{\sum_{i=1}^{n}\left[v_i\left(\sum_{j=1}^{n} x_j^2 + 1\right) - 2x_i \sum_{j=1}^{n} x_j v_j\right]^2 + 4\left(\sum_{i=1}^{n} x_i v_i\right)^2\right\} / \left(\sum_{j=1}^{n} x_j^2 + 1\right)^4$

$= 4\left(\sum_{j=1}^{n} x_j^2 + 1\right)^{-2} \|v\|^2$ So $\lambda(p) = \frac{2}{\|p\|^2 + 1}$

(b) $d\varphi(v) \cdot d\varphi(w) = \frac{1}{4}\left(\|d\varphi(v) + d\varphi(w)\|^2 - \|d\varphi(v) - d\varphi(w)\|^2\right)$ then by linearity of $d\varphi_p$