So $\|\alpha(t)\| = \|x_1\|^2 \cos t + \|u\|^2 \sin t + \langle x_1, u\rangle \sin t \cos t = \cos^2 t + \sin^2 t = 1$ So $\alpha(t) \in S$.

So far, we've found the curve. If $\sin\theta = 0$. Then $x_1 = x_2$ or $x_1 = -x_2$

If $x_1 = x_2$, done. If $x_1 = -x_2$, then find a u, s.t. $\|u\| = 1$ and $\langle x_1, u\rangle = 0$ and $\alpha(t) = x_1 \cos t + u \sin t$

$\|\alpha(t)\| = 1$, $\alpha(0) = x_1$ $\alpha(\pi) = -x_1 = x_2$ □ QED.

Note. A easier way is by using polar angular axis.

Let $x_1 = (\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \cdots, \sin\theta_1\cdots\sin\theta_{n-1}\cos\theta_n, \sin\theta_1\cdots\sin\theta_n)$

$x_2 = (\cos\theta_1', \sin\theta_1'\cos\theta_2', \sin\theta_1'\sin\theta_2'\cos\theta_3', \cdots, \sin\theta_1'\cdots\sin\theta_{n-1}'\cos\theta_n', \sin\theta_1'\cdots\sin\theta_n')$

So we just need to find a continuous curve from $(\theta_1, \cdots, \theta_n) \to (\theta_1', \cdots, \theta_n')$ in $[0, 2\pi)^n$

But $[0, 2\pi)^n$ is a convex set, so just easily find $\beta(t)$. s.t. $\beta(t) \in [0, 2\pi)^n$

$\beta(0) = (\theta_1, \cdots, \theta_n)$ $\beta(t_0) = (\theta_1', \cdots, \theta_n')$. Then define

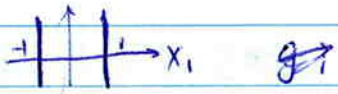
$\alpha(t) = (\cos\beta_1(t), \sin\beta_1(t)\cos\beta_1(t), \cdots, \sin\beta_1(t)\cdots\sin\beta_{n-1}(t)\cos\beta_n(t), \sin\beta_1(t)\cdots\sin\beta_n(t))$

5.2 If there exists $p, q \in S$, s.t. $g(p) = 1$, $g(q) = -1$. then

as S is connected, there exists a continuous map $\alpha: [a,b] \to S$, s.t $\alpha(a) = p$, $\alpha(b) = q$

As $g\circ\alpha$ is continous, $g(\alpha(a)) = 1$. $g(\alpha(b)) = -1$, So there exists $c \in (a,b)$, s.t. $g(\alpha(c)) = 0$

But by definition of α, $\alpha(c) \in S$ which contradicts with $g(x) = \pm 1$ for $\forall x \in S$.

5.3 1-surface: $f(x_1, x_2) = (x_1 - 1)(x_1 + 1)$ 

Define $g(x_1, x_2) = \begin{cases} -1 & x_1 \in (-3/2, -1/2) \\ 1 & x_1 \in (1/2, 3/2) \end{cases}$ So g is smooth on S, but g is not constant

5.4 $N_1(p)$ and $N_2(p)$ are both smooth. $\|\pm p/r\| = 1$, $\pm p/r \in S_p^\perp$ $\nabla(\sum_{i=1}^n x_i^2) = 2(x_1, \cdots, x_n)^T$

5.5 (a)  rotate counterclock-wise by $\pi/2$

(b) $R_\theta((v,0)) = \cos\theta \cdot (v,0) + \sin\theta \cdot (0,0,1) \times (v,0) = (v', 0)$ where $v' = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} v$

which is counterclock-wise rotation with angle θ.

(c) $\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 > 0$ So right-handed

5.6 Let θ denote the angle measured counterclock-wise from $(p, 1, 0)$ to the orientation direction $N(p)$, so that $N(p) = (p, \cos\theta, \sin\theta)$ So the positive tangent direction is $(\cos(\theta - \frac{\pi}{2}), \sin(\theta - \frac{\pi}{2})) = (\sin\theta, -\cos\theta)$. v is tangent to C at p, So $v/\|v\| = \pm(\sin\theta, -\cos\theta)$

But if $v/\|v\| = -(\sin\theta, -\cos\theta)$ then $\det\begin{pmatrix} v/\|v\| \\ N(p) \end{pmatrix} = \det\begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} = -1$. which means inconsistent

So positive tangent $\Leftrightarrow$ consistent