17.14 (a) ~~linear~~ multilinearity is obvious. We only need to prove skewsymmetry. To this end, we only need to prove for $\forall i,j \in \{1,\cdots k+l\}$, $(w_1\wedge w_2)(v_1\cdots v_i\cdots v_j\cdots v_{k+l})$ $= -(w_1\wedge w_2)(v_1\cdots v_j\cdots v_i\cdots v_{k+l})$.

For $\forall\sigma$, ~~if~~ let p,q s.t. $\sigma(p)=i, \sigma(q)=j$. If $p,q\le k$, then v_i, v_j both appear in w_1 under such σ, so swapping $v_i\ v_j$ will just inverse the sign. The same happens if $p,q>k$.

If $p\le k$, $q>k$, then look at $\hat\sigma$ which is the same as σ except $\hat\sigma(p)=j$, $\hat\sigma(q)=i$.

So $\operatorname{sign}\hat\sigma = -\operatorname{sign}\sigma$. For $(w_1\wedge w_2)(v_1\cdots v_i\cdots v_j\cdots v_{k+l})$ we have summands

$(\operatorname{sign}\sigma)\, w_1(\cdots v_i\cdots)\, w_2(\cdots v_j\cdots) - (\operatorname{sign}\sigma)\, w_1(\cdots v_j\cdots)\, w_2(\cdots v_i\cdots)$.

For $(w_1\wedge w_2)(v_1\cdots v_j\cdots v_i\cdots v_{k+l})$, we have summands

$(\operatorname{sign}\sigma)\, w_1(\cdots v_j\cdots)\, w_2(\cdots v_i\cdots) - (\operatorname{sign}\sigma)\, w_1(\cdots v_i\cdots)\, w_2(\cdots v_j\cdots)$.

So ~~for~~ the summands for swapped $v_i\ v_j$ ~~are~~ have opposite sign.

This also happens to $p>k$, $q\le k$. So in all $(w_1\wedge w_2)(v_1\cdots v_i\cdots v_j\cdots v_{k+l}) = -(w_1\wedge w_2)(v_1\cdots v_j\cdots v_i\cdots v_n)$.

(b) We only need to prove that if

$(\sigma(1)\cdots\sigma(k), \sigma(k+1), \cdots \sigma(k+l)) = (\hat\sigma(l+1)\cdots\hat\sigma(k+l), \hat\sigma(1), \cdots, \sigma(l))$, i.e.

$w_1(v_{\sigma(1)}\cdots v_{\sigma(k)})\cdot w_2(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)}) = w_2(v_{\sigma(1)}\cdots v_{\sigma(l)})\cdot w_1(v_{\sigma(l+1)}\cdots v_{\sigma(l+k)})$, then

$\operatorname{sign}\sigma = (-1)^{kl}\operatorname{sign}\hat\sigma$. This boils down to how many number of swaps is needed in order to change $(a_1\cdots a_k a_{k+1}\cdots a_{k+l})$ to $(a_{k+1}\cdots a_{k+l} a_1\cdots a_k)$, and we only care about the odd/even of the number. One schedule is pushing a_{k+1} ahead ~~for~~ by swapping with the element to its left for k times, i.e. $(a_1\cdots a_{k-1}a_k a_{k+1}) \to (a_1\cdots a_{k-1}a_{k+1}a_k) \to (a_1\cdots a_{k+1}a_{k-1}a_k) \to \cdots \to (a_{k+1}a_1\cdots a_k\cdots)$. Doing the same for $a_{k+2},\cdots a_{k+l}$, then we change $(a_1\cdots a_k a_{k+1}\cdots a_{k+l})$ to $(a_{k+1}\cdots a_{k+l} a_1\cdots a_k)$ in kl steps.

Since the odd/even of step number is independent of schedule.

we proved $\operatorname{sign}\sigma = (-1)^{kl}\operatorname{sign}\hat\sigma$.

(c) $w_1\wedge(w_2+w_3) = \frac{1}{k!\,l!}\sum_\sigma(\operatorname{sign}\sigma)\, w_1(v_{\sigma(1)}\cdots v_{\sigma(k)})(w_2+w_3)(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)})$

$= \frac{1}{k!\,l!}\sum_\sigma(\operatorname{sign}\sigma)\, w_1(v_{\sigma(1)}\cdots v_{\sigma(k)})\, w_2(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)}) + \frac{1}{k!\,l!}\sum_\sigma(\operatorname{sign}\sigma)\, w_1(v_{\sigma(1)}\cdots v_{\sigma(k)})\, w_3(v_{\sigma(k+1)}\cdots v_{\sigma(k+l)})$

$= (w_1\wedge w_2) + (w_1\wedge w_3)$.

(d) $(w_1\wedge w_2)\wedge w_3 = \frac{1}{k!\,l!\,m!\,(k+l)!}\sum_{\sigma,\hat\sigma}(\operatorname{sign}\sigma)(\operatorname{sign}\hat\sigma)\, w_1(v_{\sigma(\hat\sigma(1))},\cdots v_{\sigma(\hat\sigma(k))})\, w_2(v_{\sigma(\hat\sigma(k+1))}\cdots v_{\sigma(\hat\sigma(k+l))})\, w_3(v_{\sigma(k+l+1)}\cdots v_{\sigma(k+l+m)})$ (*)

where σ is a permutation of $1\cdots(k+l+m)$ and $\hat\sigma$ is a permutation of $1\cdots k+l$.

Notice $(\operatorname{sign}\sigma)\cdot(\operatorname{sign}\hat\sigma) = \operatorname{sign}(\sigma\circ\hat\sigma)$. (we can define $\hat\sigma(i)=i$ for $i>k+l$).

For each $w_1(v_{i_1}\cdots v_{i_k})\, w_2(v_{i_{k+1}}\cdots v_{i_{k+l}})\, w_3(v_{i_{k+l+1}}\cdots v_{i_{k+l+m}})$, there exist $(k+l)!$ different combinations of σ and $\hat\sigma$ which finally results in this order of subscript by permutating from $(1,\cdots,k+l+m)$. In fact, for any $\hat\sigma$, there exists a unique σ, such that $\sigma\circ\hat\sigma$ yields above subscripts. Besides, all such combinations have the same sign of $\sigma\circ\hat\sigma$. So (*) is