22.1 Example 1: $\|\psi(p)-\psi(q)\| = \|p+a-(q+a)\| = \|p-q\|$

Example 2: $\|\psi(p)-\psi(q)\| = \|Ap-Aq\| = \|A(p-q)\| = \|p-q\|$, as $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$. $\|A\begin{pmatrix}x_1\\x_2\end{pmatrix}\| =$
$\|(\cos\theta x_1 - \sin\theta x_2, \sin\theta x_1 + \cos\theta x_2)\| = [(\cos\theta x_1 - \sin\theta x_2)^2 + (\sin\theta x_1 + \cos\theta x_2)]^{1/2} = (x_1^2+x_2^2)^{1/2}$

Example 3: $\|\psi(p)-\psi(q)\| = \| p+2(b-p\cdot a)\cdot a - q - 2(b-q\cdot a)a\| = \|p-q-2[(p-q)\cdot a]\cdot a\|$, let $x=p-q$
$=[(x-2(x\cdot a)\cdot a)^T(x-2(x\cdot a)\cdot a)]^{1/2} = [x^Tx + 4(x\cdot a)^2 - 4(x\cdot a)^2]^{1/2} = \|x\| = \|p-q\|$.

22.2. $\forall x\in R^{n+1}$, $\psi_1(\psi_2(x)) = \psi_1(x+a) \overset{\psi_1 \text{ is linear}}{=} \psi_1(x)+\psi_1(a) = \tilde{\psi}_2(\psi_1(x))$, $\tilde{\psi}_2(\hat{x}) = \hat{x}+\psi_1(a)$.

22.3(a) $\psi(v)\cdot\psi(w) = v\cdot w \Rightarrow \psi(v)\cdot\psi(v) = v\cdot v \Rightarrow \|\psi(v)\| = \|v\|$.
$\|\psi(v)\| = \|v\| \Rightarrow \psi(v)\cdot\psi(w) = \frac{1}{2}[\|\psi(v+w)\|^2 - \|\psi(v)\|^2 - \|\psi(w)\|^2] =$
$= \frac{1}{2}[\|v+w\|^2 - \|v\|^2 - \|w\|^2] = v\cdot w$

(b) ① $\forall$ orthonormal basis $\{e_1\cdots e_n\}$. let $v=\sum_{i=1}^n v_i e_i$, then if $\{\psi(e_1)\cdots\psi(e_{n+1})\}$ is orthonormal
we have $\|\psi(v)\| = \|\psi(\sum_{i=1}^n v_i e_i)\| = \|\sum_{i=1}^n v_i\psi(e_i)\| = \sqrt{\sum_{i=1}^n v_i^2} = \|v\|$

② By (a), If $\{e_1\cdots e_n\}$ is orthonormal, then $\psi(e_i)\cdot\psi(e_j) = e_i\cdot e_j = \begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$
so $\{\psi(e_1),\cdots,\psi(e_n)\}$ is orthonormal basis for R^{n+1}

(c) ⊕ Let $\psi(e_i) = \sum_{j=1}^n a_{ij}e_j$, where $\{e_1\cdots e_n\}$ is an orthonormal basis of R^{n+1}.
① If A is orthogonal, then letting $P=(\psi(e_1),\cdots,\psi(e_n)) = AQ$ where $Q=(e_1\cdots e_n)$,
we have $P^T\cdot P = Q^TA^TAQ = Q^TQ = I$, so $\psi(e_1)\cdots\psi(e_n)$ is orthonormal
By (b) we have ψ is orthogonal transformation.
② If ψ is orthogonal, then by (b) $P=(\psi(e_1)\cdots\psi(e_n))$ is also orthonormal
$I = P'P = AQQ^TA^T = AA^T$ So A is orthogonal.

22.4 (a) By Ex 22.3(c). The matrix is orthonormal $\Leftrightarrow$ orthogonal linear transformation
So rotation $\Leftrightarrow \begin{vmatrix}x_1 & x_3\\ x_2 & x_4\end{vmatrix} = 1$ and $A^TA=I$ where $A=\begin{pmatrix}x_1 & x_2\\ x_2 & x_4\end{pmatrix}$.
$\Leftrightarrow x_1^2+x_3^2=1,\ x_1x_2+x_3x_4=0,\ x_2^2+x_4^2=1,\ x_1x_4-x_2x_3=1$ (*)
Let $x_1=\cos\theta$, $x_3=-\sin\theta$, $x_2=\cos\varphi$, $x_4=\sin\varphi$, we have
$\cos(\theta+\varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi = x_1x_2+x_3x_4 = 0$
$\sin(\theta+\varphi) = \sin\theta\cos\varphi + \cos\theta\sin\varphi = -x_3x_2+x_1x_4 = +1$
So $\theta+\varphi = 2k\pi + \frac{\pi}{2}$, $\sin\varphi = \cos\theta$, $\cos\varphi = \sin\theta$, so $A=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$
obviously, A in such a form must satisfy (*)

(b) $\forall$ eigenvalue and eigenvector α_i, λ_i: $\psi\alpha_i = \lambda_i\alpha_i$, then $\alpha_i^T\psi^T\psi\alpha_i = \alpha_i^T\lambda_i^2\alpha_i$
As $\psi^T\psi = I$ by Ex 22.3(c), $1 = \alpha_i^T\alpha_i = \lambda_i^2$. So $\lambda_i = \pm1$. If all λ_i are -1
then $|\psi| = \prod_{i=1}^3\lambda_i = -1$, violating definition of rotation. So $\exists_{\lambda_i=1}\alpha_i$: $\psi\alpha_i = \alpha_i$.