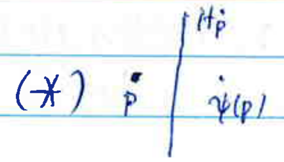
(c) For $\forall v \perp e_1$, $\psi(v)\cdot\psi(e_1) = v\cdot e_1 = 0$, So $v \perp e_i$, So the matrix must be in the form of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1 & x_3 \\ 0 & x_2 & x_4 \end{pmatrix}$. A orthonormal $\Leftrightarrow$ $\begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}$ orthonormal, $\begin{vmatrix} x_1 & x_3 \\ x_2 & x_4 \end{vmatrix} = |A| = 1$.
So by the proof in (a), $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$.

22.5 Map: $\forall (x_1, x_2)$ on $x_1 x_2 = 1$ to $\varphi(x_1, x_2) = \left(\frac{\sqrt{2}}{2}(x_1+x_2), \frac{\sqrt{2}}{2}(x_1 - x_2)\right)$.
Obviously, $\varphi(x_1, x_2)$ is on $x_1^2 - x_2^2 = 2$. $\|\varphi(x_1,x_2) - \varphi(x_1', x_2')\| =$
$\left\|\left(\frac{\sqrt{2}}{2}(x_1+x_2-x_1'-x_2'), \frac{\sqrt{2}}{2}(x_1-x_2-x_1'+x_2')\right)\right\| = \sqrt{\frac{1}{2}(m+n)^2 + \frac{1}{2}(m-n)^2}$ $(m \triangleq x_1 - x_1',\ n \triangleq x_2 - x_2')$
$= \sqrt{m^2+n^2} = \|(x_1 - x_1', x_2 - x_2')\|$ So $\varphi$ is rigid motion.

22.6 (a) $0 = \|x - \psi(p)\|^2 - \|x-p\|^2 = 2(p - \psi(p))\cdot x + \|\psi(p)\|^2 - \|p\|^2$ $(*)$




As $p \notin F$, so $p - \psi(p) \neq 0$ So $H_p$ is hyperplane

(b) $\forall q \in F$, $\|q - \psi(p)\| = \|\psi(q) - \psi(p)\| = \|q - p\|$. So $q \in H_p$, so $F \subseteq H_p$.

(c) ~~By $(*)$ in (a), $p - \psi(p) \perp H_p$~~. Obviously, $q = \frac{1}{2}(\psi(p) + p) \in H_p$
By $(*)$ $q - p = \frac{1}{2}(\psi(p) - p) \perp H_p$, $q - \psi(p) = \frac{1}{2}(p - \psi(p)) \perp H_p$.
So the line segment $p \to \psi(p)$ intersects with $H_p$ perpendicularly at $q$.
As $\|q-p\| = \|q - \psi(p)\|$, $\psi_p(\psi(p)) = p$ i.e. p is fixed point of $\psi_p \circ \psi$.
Besides, as $F \subset H_p$ and $\psi(F) \subseteq F$ and $\psi_p$ is reflection through $H_p$,
it is obvious that $F \subseteq \{$fixed point of $\psi_p \circ \psi\}$

(d) Suppose $\psi = \psi_2 \circ \psi_1$ is the unique decomposition of $\psi$ into an orthogonal transformation $\psi_1$ followed by translation $\psi_2$. As $\psi(0) = 0$, $\psi_2$ is identity. So
$\psi\left(\sum_{i=1}^{k} c_i p_i\right) \underset{\text{as } \psi_2 = \text{identity}}{=} \psi_1\left(\sum_{i=1}^{k} c_i p_i\right) \underset{\text{linearity of } \psi_1}{=} \sum_{i=1}^{k} c_i \psi_1(p_i) \underset{\psi_2 \text{ is identity}}{=} \sum_{i=1}^{k} c_i \psi(p_i) \underset{\text{as } p_i \in F}{=} \sum_{i=1}^{k} c_i p_i$, so $\sum_{i=1}^{k} c_i p_i \in F$.

(e) Denote $\varphi_0 = \psi$, ~~$\varphi_i = \psi_{e_i} \circ \varphi_{i-1}$ for $i = 1, \cdots, n+1$ where $e_i$ are standard bases of $R^{n+1}$~~
~~We prove by~~ Let $e_1 \cdots e_{n+1}$ be the standard bases of $R^{n+1}$
If ~~all~~ $0 \notin F$, then denote $\varphi_0 = \psi_0 \circ \psi$, $F_0 =$ the set of fixed points of $\psi_0 \circ \psi$. By (c) $0 \in F_0$. If $0 \in F$, then $\varphi_0 \triangleq \psi$, $F_0 = F$.
If $e_1 \notin F_0$, then denote $\varphi_1 = \psi_{e_1} \circ \psi_0$, $F_1 =$ the set of fixed points of $\varphi_1$.
By (c) $e_1 \in F_1$, $F_0 \subset F_1$ so $0 \in F_1$
The same procedure goes on, until $e_{n+1}$. Then $e_i \in F_{n+1}$, $i = 1 \cdots n+1$, $0 \in F_{n+1}$.
By (d) $\sum_{i=1}^{n+1} c_i p_i \in F_{n+1}$ whenever $p_1 \cdots p_{n+1} \in F$, $c_i \in R$. So $F_{n+1} = R^{n+1}$. This means $\varphi_{n+1}$ is identity, i.e. there exists a $k \leq n+2$, and reflections $\psi_1 \cdots \psi_k$ of $R^{n+1}$ s.t. $\psi_k \circ \cdots \circ \psi_1 \circ \psi = I$. As reflections are all invertible and its inversion is itself, So $\psi = \psi_1^{-1} \circ \cdots \psi_k^{-1} = \psi_1 \circ \cdots \circ \psi_k$.