A is like $\begin{pmatrix}\beta_{11}&\beta_{12}&0\\\beta_{21}&\beta_{22}&0\\\beta_{31}&\beta_{32}&\neq 0\end{pmatrix}$. But as $\begin{pmatrix}\beta_{11}\\\beta_{12}\end{pmatrix}\perp\begin{pmatrix}\beta_{21}\\\beta_{22}\end{pmatrix}$, it is impossible for $\begin{pmatrix}\beta_{31}\\\beta_{32}\end{pmatrix}$ to be orthogonal to both $\begin{pmatrix}\beta_{11}\\\beta_{12}\end{pmatrix}$ and $\begin{pmatrix}\beta_{21}\\\beta_{22}\end{pmatrix}$, unless $\begin{pmatrix}\beta_{31}\\\beta_{32}\end{pmatrix}=0$. Thus $\beta_{33}=\pm1$. In sum $A=\begin{pmatrix}\beta_{11}&\beta_{12}&0\\\beta_{21}&\beta_{22}&0\\0&0&\pm1\end{pmatrix}$. Finally the ~~possible~~ symmetric group of cylinder $x_1^2+x_2^2=a^2$ in R^3 is $\varphi(P_1,P_2,P_3)=(\beta_{11}P_1+\beta_{12}P_2,\ \beta_{21}P_1+\beta_{22}P_2,\ \nu P_3+a_3)$, where $\nu=1$ or -1, $a_3\in R$, $\begin{pmatrix}\beta_{11}&\beta_{12}\\\beta_{21}&\beta_{22}\end{pmatrix}$ is orthonormal.

The discussion above has shown that the above conditions are both necessary and sufficient.

(d) Using same notation as in (c), $\frac{1}{a^2}(\varphi_1(P)+a_1)^2+\frac{1}{b^2}(\varphi_2(P)+a_2)^2+\frac{1}{c^2}(\varphi_3(P)+a_3)^2=1$

$\frac{1}{a^2}(-\varphi_1(P)+a_1)^2+\frac{1}{b^2}(-\varphi_2(P)+a_2)^2+\frac{1}{c^2}(-\varphi_3(P)+a_3)^2=1$, So $\frac{a_1}{a^2}\varphi_1(P)+\frac{a_2}{b^2}\varphi_2(P)+\frac{a_3}{c^2}\varphi_3(P)=0$ (*)

As φ is onto, there must be a p_0 on this ellipsoid S, s.t.

$\varphi(P_0)=(\varphi_1(P_0)+a_1,\ \varphi_2(P_0)+a_2,\ \varphi_3(P_0+a_3))=(-a_1a,-a_2b,-a_3c)/r$

where $r=(a_1^2a^2+b^2a_2^2+c^2a_3^2)^{1/2}$. Assume now $r\neq0$.

Then $\frac{a_1}{a^2}\varphi_1(P)+\frac{a_2}{b^2}\varphi_2(P)+\frac{a_3}{c^2}\varphi_3(P)=-\frac{1}{r}\left(\frac{a_1^2}{a}+\frac{a_2^2}{b}+\frac{a_3^2}{c}\right)-\left(\frac{a_1^2}{a^2}+\frac{a_2^2}{b^2}+\frac{a_3^2}{c^2}\right)<0$, contradicting (*)

So we must have $r=0$, i.e. $a_1=a_2=a_3=0$.

(ii) If a,b,c are distinct, then w.l.o.g. assume $c<b$, $c<a$. Consider point $(0,0,c)$ on S. $\varphi(0,0,c)=(c\beta_{13},c\beta_{23},c\beta_{33})$, If it is ~~which~~ ~~should~~ ~~must~~ be on S, ~~thus~~ then

$1=\frac{c^2\beta_{13}^2}{a^2}+\frac{c^2\beta_{23}^2}{b^2}+\frac{c^2\beta_{33}^2}{c^2}\le\frac{c^2}{c^2}(\beta_{13}^2+\beta_{23}^2+\beta_{33}^2)=1$. ~~So the symmetry group of S is empty.~~

~~(i) If $a<b=c$, then same logic as above. Otherwise consider point $(a,0,0)$~~
~~$\varphi(a,0,0)=(a\beta_{11},a\beta_{21},a\beta_{31})$. If it is on S, then~~
~~$1=\frac{a^2\beta_{11}^2}{a^2}+\frac{1}{b^2}a^2\beta_{21}^2+\frac{1}{c^2}a^2\beta_{31}^2>\frac{a^2}{a^2}(\beta_{11}^2+\beta_{21}^2+\beta_{31}^2)=1$. So still empty is the symmetry group of S.~~

The equality holds iff $\beta_{23}=\beta_{33}=0$ So $\beta_{13}=\pm1$. Similarly $\beta_{21}=\beta_{31}=0$ $\beta_{11}=\pm1$

So A is like $\begin{pmatrix}\pm1&\beta_{12}&0\\0&\beta_{22}&0\\0&\beta_{32}&\pm1\end{pmatrix}$. so $A=\begin{pmatrix}\pm1&0&0\\0&\pm1&0\\0&0&\pm1\end{pmatrix}$. So the symmetry group of S is $\varphi(P_1,P_2,P_3)=(\delta_1P_1,\delta_2P_2,\delta_3P_3)$ where $\delta_i=\pm1$ $i=1,2,3$.

(i) If $b=c$, $a\neq b$, then as in (ii) we have $\beta_{21}=\beta_{31}=0$. Besides, as $(\beta_{12}b,\beta_{22}b,\beta_{32}b)$ is on S, we have $1=\frac{b^2\beta_{12}^2}{a^2}+\frac{b^2\beta_{22}^2}{b^2}+\frac{b^2\beta_{32}^2}{c^2}\ge\frac{b^2}{b^2}(\beta_{12}^2+\beta_{22}^2+\beta_{32}^2)=1$

Equality hold iff $\beta_{12}=0$. Likewise $\beta_{13}=0$. So A is like $\begin{pmatrix}\pm1&0&0\\0&\beta_{22}&\beta_{23}\\0&\beta_{32}&\beta_{33}\end{pmatrix}$.

A is orthonormal $\Rightarrow$ $\begin{pmatrix}\beta_{22}&\beta_{23}\\\beta_{32}&\beta_{33}\end{pmatrix}$ is orthonormal. Conversely $\begin{pmatrix}\beta_{22}&\beta_{23}\\\beta_{32}&\beta_{33}\end{pmatrix}$ being orthonormal is sufficient because $\varphi(P_1,P_2,P_3)=(\pm P_1,\ \beta_{22}P_2+\beta_{23}P_3,\ \beta_{32}P_2+\beta_{33}P_3)$ and $\frac{1}{b^2}(\beta_{22}P_2+\beta_{23}P_3)^2+\frac{1}{c^2}(\beta_{32}P_2+\beta_{33}P_3)^2=\frac{1}{b^2}[P_2^2+P_3^2]$, So $\varphi(P_1,P_2,P_3)\in S$ and obviously $(P_2,P_3)^T\to\begin{pmatrix}\beta_{22}&\beta_{23}\\\beta_{32}&\beta_{33}\end{pmatrix}\begin{pmatrix}P_2\\P_3\end{pmatrix}$ is invertible and bijective from ~~S to S~~ $P_2^2+P_3^2=b^2(1-\frac{P_1^2}{a^2})$ to itself. Thus the symmetry group of S is $\varphi(P_1,P_2,P_3)=\begin{pmatrix}\pm1&0&0\\0&\beta_{22}&\beta_{23}\\0&\beta_{32}&\beta_{33}\end{pmatrix}\begin{pmatrix}P_1\\P_2\\P_3\end{pmatrix}$ where $\begin{pmatrix}\beta_{22}&\beta_{23}\\\beta_{32}&\beta_{33}\end{pmatrix}$ is orthonormal