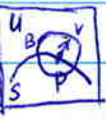
6.6 Obvious ?

6.7 "if part": Suppose the orientation at p is $N(p)$. Since $\alpha(t) = p + ta \in S$ for all $t \in I$, so $\dot{\alpha}(0) = a \in S_p$. So $a \cdot N(p) = 0$ which is true for any $p \in S$.

"only if" part: Consider the constant vector field $X(q) = (q, a)$. It is a tangent field on S because at $\forall p$. $N(p) \cdot a = 0$. Now $\alpha(t) = p + at$ is an integral curve of X and $\alpha(0) = p \in S$. Then by the corollary to Theorem 1, Chapter 5, $\alpha(t) \in S$ for all $t \in I$ where I is the interval on which $\alpha(t)$ is defined.



6.8 Suppose $N(S) = \{v\}$. Let B be an open ball contained in U (S is a level set on U) and $p \in S \cap B$. Then for $\forall x_0 \in B$ which satisfies $(x_0 - p) \cdot v = 0$, we construct a constant vector field $W(q) = (q, x_0 - p)$, ~~which is the those~~ Since $N(S) = \{v\}$, the restriction of $W(q)$ on U is a tangent vector field on S. As B is open, there is $\alpha(t) = p + (x_0 - p)t$ ($(-\varepsilon, 1+\varepsilon) \to B$), an integral curve of W, such that $\alpha(0) \in S$. Thus by corollary to Thm 1, ch 5, $\alpha(t) \in S$, and specifically $\alpha(1) = x_0 \in S$. Therefore $\{x \in \mathbb{R}^{n+1} : x \cdot v = p \cdot v\} \cap B \subseteq S$.

Next, suppose a new $\alpha : [a, b] \to S$ is a continuous parametrized curve and $\alpha(t) \in B'$ (a new open set) for $t_1 \le t \le t_2$.
If $\alpha(t_1) \cdot v < \alpha(t_2) \cdot v$, then for any $b \in (\alpha(t_1) \cdot v, \alpha(t_2) \cdot v)$, due to $\alpha(t)$ being continuous, there exists $t_3 \in (t_1, t_2)$ s.t. $\alpha(t_3) \cdot v = b$. Since $\alpha(t_3) \in S \cap B$, by above argument, we have $\{x \in \mathbb{R}^{n+1} : x \cdot v = \alpha(t_3) \cdot v = b\} \cap B \subseteq S$. Therefore $\{x \in B \mid x \cdot v \in (\alpha(t_1) \cdot v, \alpha(t_2) \cdot v)\} \subseteq S$. But ~~that~~ the left hand set is an open set and therefore $N(S) = S^n$ (because S contains an open set), contradicting with $N(S) = \{v\}$. So $\alpha(t_1) \cdot v \ge \alpha(t_2) \cdot v$. Likewise, $\alpha(t_1) \cdot v \le \alpha(t_2) \cdot v$.
So $\alpha(t_1) \cdot v = \alpha(t_2) \cdot v$. Since S is connected, any two points on S p, q can be connected by a continuous parametrized curve, and one can find an open set B s.t. $p, q \in S \cap B$, therefore $p \cdot v = q \cdot v$, i.e., all points in S lie on the same plane (or part of a plane).

6.9 (a) Let $g(t) = f(\alpha(t))$. So now we have $g(t_1) = g(t_2) = c$ $g(t) \ne c$ for all $t \in (t_1, t_2)$
If $g'(t_1) > 0$, $g'(t_2) > 0$. Then there exists $\varepsilon_1 > 0$, s.t. $g'(t) > 0$ $t \in [t_1, \varepsilon_1 + t_1)$
then $g(t_1 + \frac{\varepsilon_1}{2}) - g(t_1) = g'(t_1 + \xi_1) \cdot \frac{\varepsilon_1}{2}$ where $\xi_1 \in [0, \frac{\varepsilon_1}{2}]$. So $g'(t_1 + \xi_1) > 0$,
thus $g(t_1 + \frac{\varepsilon_1}{2}) > g(t_1) = c$ There also exists $\varepsilon_2 > 0$ s.t. $g'(t) > 0$ $t \in (t_2 - \varepsilon_2, t_2]$
then $g(t_2 - \frac{\varepsilon_2}{2}) - g(t_2) = -g'(t_2 - \xi_2)\frac{\varepsilon_2}{2}$, where $\xi_2 \in [0, \frac{\varepsilon_2}{2}]$ so $g'(t_2 - \xi_2) > 0$
thus $g(t_2 - \frac{\varepsilon_2}{2}) < g(t_2) = c$. Then as g is continuous there exists $t \in (t_1 + \frac{\varepsilon_1}{2}, t_2 - \frac{\varepsilon_2}{2}) \subset (t_1, t_2)$
s.t. $g(t) = c$ contraction!

one can choose small enough $\varepsilon_1, \varepsilon_2$, s.t. $t_1 + \frac{\varepsilon_1}{2} < t_2 - \frac{\varepsilon_2}{2}$