$\alpha_3(b)=\alpha_2(t_2)$. $\alpha_3(t)\in O(S)$ by (a). So now construct ~~a~~ continuous curve from p to q in $O(S)$:

$$\gamma(t)=\begin{cases}\alpha_1(t) & t\in[0,t_1]\\ \alpha_3(t-t_1+a) & t\in[t_1,t_1+b-a]\\ \alpha_2(t_2-t+t_1+b-a) & t\in(t_1+b-a,\ t_1+b-a+t_2)\end{cases}$$

7.2 $\|\dot\alpha(t)\|=\text{constant} \Rightarrow \frac{d}{dt}\dot\alpha(t)\cdot\dot\alpha(t)=2\ddot\alpha(t)\cdot\dot\alpha(t)=0$, i.e. $\ddot\alpha(t)\perp\dot\alpha(t)$

7.3 Let $s(t)=\int_{t_0}^{t}\|\dot\alpha(t)\|dt$. As $\dot\alpha(t)\neq 0$, so $s(t)$ monotonic increasing so $s(t)$ is invertible. Let $h=s^{-1}$. h is onto by definition $h'=\frac{1}{s'}=\|\dot\alpha(h(t))\|^{-1}>0$

$\dot\beta=\dot\alpha(h(t))\cdot h'(t)=\dot\alpha(h(t))/\|\dot\alpha(h(t))\|$ so β is unit speed

7.5. ~~$\dot\alpha$ for~~ "if part" is by Example 2 in this chapter

"only if". $\alpha(0)=(r\cos b, r\sin b, d)$, which has covered all possible points on cylinder

$\dot\alpha(0)=(-ra\sin b, ra\cos b, c)$. $N_{\alpha(0)}=\pm(\cos b,\sin b, 0)$.

So $\dot\alpha(0)$ has covered all possible initial velocity in $S_{\alpha(0)}$

As geodesic is uniquely determined by initial position and initial velocity these are all possible geodesics on cylinder S.

Another proof is by looking at (G) on page 41. $N(x,y,z)=(x,\hat{y},0)$

7.6 "if part" is covered by Example 3 in this chapter

"only if" $\alpha(0)=e_1$, $\dot\alpha(0)=ae_2$. Since $e_2\in S_{e_1}$. a allows all norm of velocity

e_1 allows all possible initial position, $\dot\alpha(0)$ allows all possible initial velocity

due to uniqueness of geodesic by initial position and velocity, these are all possible ~~de~~ geodesics on unit n-sphere.

7.7 "if part": $\dot\beta(t)=\dot\alpha(h(t))h'(t)=\dot\alpha(at+b)\cdot a$ $\ddot\beta(t)=a^2\ddot\alpha(at+b)$ As $\alpha(t)$ is geodesic so $\ddot\alpha(t)\in S^{\perp}_{\alpha(t)}\ \forall t$. So $\ddot\beta(t)\in S^{\perp}_{\alpha(at+b)}=S^{\perp}_{\beta(t)}$ So β is geodesic

"only if": $\ddot\beta(t)=\ddot\alpha(h(t))\cdot(h'(t))^2+\dot\alpha(h(t))h''(t)$ if β is geodesic, $\ddot\beta(t)\in S^{\perp}_{\beta(t)}=S^{\perp}_{\alpha(h(t))}\ni\ddot\alpha(h(t))$

So $\ddot\beta(t)$ and $\ddot\alpha(h(t))$ are parallel generally. $\dot\alpha$ and $\ddot\alpha$ are not parallel, and $h''(t)$, $h'(t)$ are scalar

So we must require $h''(t)=0$ (E.g. $\alpha(t)=\hat{e}_1\cos t+\hat{e}_2\sin t$ $\dot\alpha(t)=-\hat{e}_1\sin t+\hat{e}_2\cos t$

$\ddot\alpha(t)=-\hat{e}_1\cos t-\hat{e}_2\sin t$, $\theta_{\dot\alpha,\ddot\alpha}=0$. So $\dot\alpha$ and $\ddot\alpha$ are never parallel).

So $h(t)=at+b$. We can't see why we need $a>0$. Since at least when $a=0$. β is still geodesic