7.8 (a) $\dot\alpha_\theta(t) = (\dot x_1(t),\ \dot x_2(t)\cos\theta,\ \dot x_2(t)\sin\theta)$

$\dot\beta_t(\theta) = (0, -x_2(t)\sin\theta,\ x_2(t)\cos\theta)$

$\dot\alpha_\theta(t)\cdot\dot\beta_t(\theta) = 0$

(b) $\ddot\alpha_\theta(t) = (\ddot x_1(t),\ \ddot x_2(t)\cos\theta,\ \ddot x_2(t)\sin\theta)$

~~Sp~~ $N(p) = \pm ?$ hard to write So must find another way

Notice that $\dot\alpha_\theta(t) \in S_p$. $\dot\beta_t(\theta) \in S_p$ by definition because $\alpha_\theta(t), \beta_t(\theta)$ are both on S.

by (a) $\dot\alpha_\theta(t) \perp \dot\beta_t(\theta)$. So $\dot\alpha_\theta(t), \dot\beta_t(\theta)$ form a basis of S_p $(p = \alpha_\theta(t))$

So one only needs to check that $\ddot\alpha_\theta(t)$ is orthogonal to $\dot\alpha_\theta(t)$ and $\dot\beta_t(\theta)$

$\cancel{\dot\alpha_\theta(t)\cdot\ddot\alpha_\theta(t) = \ddot x_1(t)\dot x_1(t) + \ddot x_2(t)\dot x_2(t)}$. As $\alpha(t) = (x_1(t), x_2(t))$ has constant speed, by Ex 7.2 $\dot\alpha_\theta(t) \perp \ddot\alpha_\theta(t)$. $\ddot\alpha_\theta(t) \perp \dot\beta_t(\theta)$ is easy to check.

(c) $\ddot\beta_t(\theta) = (0, -x_2(t)\cos\theta, -x_2(t)\sin\theta)$, obviously $\ddot\beta_t(\theta) \perp \dot\beta_t(\theta)$

$\ddot\beta_t(\theta) \perp \dot\alpha_\theta(t) \Leftrightarrow x_2(t)\cdot\dot x_2(t) = 0$ Since $x_2(t) > 0$ $\dot x_2(t) = 0 \Leftrightarrow x_2'(t)/x_1'(t)\big|_{\alpha(t)} = 0$

7.9 First check $\beta(t) = \alpha(ct)$ is a maximal ~~d~~ geodesic with initial velocity cv: $\beta(0) = \alpha(0)$

$\dot\alpha(ct) = c\cdot\dot\alpha(t)$ ~~ct~~. So $\dot\alpha(ct)|_{t=0} = c\cdot\dot\alpha(t)|_{t=0} = cv$.

$\ddot\beta(t) = c^2\ddot\alpha(ct)$. As α is geodesic, so $\ddot\alpha(t) \in S^\perp_{\alpha(t)}$. So $\ddot\beta(t) \in S^\perp_{\alpha(ct)} = S^\perp_{\beta(t)}$

So $\beta(t)$ is geodesic. ~~I is easily~~ Since the geodesic with given initial position and velocity ~~given~~ is unique, $\beta(t)$ is ~~what w~~ the maximal geodesic in S with initial velocity cv.

The domain I can be easily taken care of.

7.10 Define $\gamma(t) = \beta(t+t_0)$, then $\gamma(0) = \beta(t_0) = p$, $\dot\gamma(0) = \dot\beta(t_0) = v$. So if $\gamma(t)$ is geodesic, then by uniqueness theorem, $\gamma(t) = \alpha(t)$, i.e. $\beta(t+t_0) = \alpha(t)$, i.e. $\beta(t) = \alpha(t-t_0)$

I is taken care of because α is maximal

7.11 Let $\gamma(t) = \beta(t)$. $\gamma(t_0) = \beta(t_0) = \beta(0)$ ~~$= \gamma(t_0)$~~, $\dot\gamma(t_0) = \dot\beta(t_0) = \dot\beta(0)$. So by Ex 7.10

$\gamma(t) = \beta(t-t_0)$ i.e. $\beta(t) = \beta(t-t_0)$ i.e. $\beta(t+t_0) = \beta(t)$

7.12 (a) complete by Example 3

(b) incomplete $\alpha(t) = (1,0,\cdots,0)\cos t + (0,\cdots,0,1)\sin t$ is geodesic $t \in (-\frac{3\pi}{2}, \frac{\pi}{2})$ but $t \neq \frac{\pi}{2} + 2k\pi$ $k \in \mathbb{Z}$

(c) incomplete $\alpha(t) = (0,1,1) - (0,1,1)t$ $t \neq 1$

(d) complete by Example 2

(e) incomplete $\alpha(t) = (0,1,0)\cos t + (1,0,0)\sin t$ $t \neq \frac{\pi}{2} \pm 2k\pi$ $K \in \mathbb{Z}$